On the Shooting Method Applied to Richards' Equation with a Forcing Term

Fabio Vito Difonzo[1(✉)] and Giovanni Girardi[2]

[1] Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy
fabio.difonzo@uniba.it
[2] Istituto di Ricerca sulle Acque, Consiglio Nazionale delle Ricerche, Italy, Via F. De Blasio 5, 70132 Bari, Italy
giovanni.girardi@ba.irsa.cnr.it
https://sites.google.com/site/fabiovdifonzo/home

Abstract. The problem of modeling water flow in the root zone with plant root absorption is of crucial importance in many environmental and agricultural issues, and is still of interest in the applied mathematics community. In this work we propose a formal justification and a theoretical background of a recently introduced numerical approach, based on the shooting method, for integrating the unsaturated flow equation with a sink term accounting for the root water uptake model. Moreover, we provide various numerical simulations for this method, comparing the results with the numerical solutions obtained by MATLAB `pdepe`.

Keywords: Numerical simulations · Richards' equation · Shooting method

1 Introduction

The problem of water movement in the root zone needs attention in different scientific areas, and with different tools. For instance, this problem is interesting for applied mathematicians, hydrogeologists, agronomists, and numerical analysis scientists. As a matter of fact, mainly when considering the unsaturated condition (which is exactly the case of subsurface and vadose zone applications) the problem becomes challenging from different areas of mathematical research. On the other hand, it is still open the interest in characterizing soils from hydraulic point of view, with laboratory experiments, coupled sometimes with data driven models for approximating hydraulic functions (see for instance [1]).

In the wide range of applications for this physical process, agronomical applications play a primary role, since the understanding of water dynamics into the soil is necessary for efficiently managing any irrigation process (see for instance [2–4]).

The usual tool for modeling the water flow throughout an unsaturated porous medium is the Richards' equation, a parabolic PDE which still deserves a significant research attention (see, for instance, [5–7]). Such a standard infiltration

© Springer Nature Switzerland AG 2021
O. Gervasi et al. (Eds.): ICCSA 2021, LNCS 12949, pp. 273–286, 2021.
https://doi.org/10.1007/978-3-030-86653-2_20

problem is further complicated by possible fractures in the porous media (see [8,9]); or by soil heterogeneities, which can be treated by well-established discontinuous numerical approaches (see for instance [10–12]), giving rise to successful techniques in 1D and 2D spatial domains ([13,14]).

This model could be even endowed with a sink term representing (for instance) the root water uptake (e.g. [15–20]). In this work we provide a theoretical background for a new approach, first proposed in [15]. Such an approach can manage the choice of Gardner's constitutive relations in solving Richards' equation, differently from [21]; can take into account possibly any selection of root uptake functions; and finally, no variable rescaling is necessary, as in [22].

We consider the θ-form of Richards' equation in a one-dimensional spatial domain:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{\mathrm{d}K(\theta)}{\mathrm{d}\theta}\frac{\partial \theta}{\partial z} - R(\theta, t, z), \quad t \in [0, T],\ z \in [0, Z], \tag{1}$$

where θ represents the volumetric water content, R is the root-water uptake term, D is the soil water diffusivity and K is the hydraulic conductivity (see [22]); $[0, T]$ is the time domain and $[0, Z]$ is the spatial domain. The soil-water diffusivity is

$$D(\theta) := K(\theta)\frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta}; \tag{2}$$

as usual h is the suction head in the unsaturated zone, and the plant-root extraction function $-R \in (-R_s, 0]$, with $R_s \geq 0$, depends on $\theta \in [\theta_r, \theta_S]$, where θ_r is the residual water content and θ_s is the water content at saturation, assuming that z is positive downward. As in [22], it is assumed that R has no explicit dependence on time t (differently from [23]), "despite root-water uptake is, in many cases, driven by daily cycles". It is also assumed that $R'(\theta) > 0$ and $R''(\theta) < 0$.

The function R has generally a sigmoid shape, common to many other biological context (see for instance [24]): a classical piecewise linear model for root water uptake function was proposed in [25] and, more recently, a stepwise approach has been introduced in [26], where stationary solutions are studied for Richards' equation with discontinuous root water uptake models. In [27], different piecewise Feddes uptake functions are considered, according to the depth; in [28] different uptake functions are reviewed, whereas in [29] the use of root uptake functions is discussed when in presence of water with high salinity.

2 The *Shooting Method* Approach and Its Theoretical Background

We now endow (1) with initial and Dirichlet boundary conditions, which characterize water contents at the top and bottom of the domain. In particular, let $\theta^0 : [0, Z] \to [0, +\infty)$ be the initial state profile, $\theta_0 : [0, T] \to [0, +\infty]$ and $\theta_Z : [0, T] \to [0, +\infty]$ be sufficiently smooth boundary functions, and let us assume that

$$\theta(0,z) = \theta^0(z), \quad z \in [0,Z], \tag{3a}$$
$$\theta(t,0) = \theta_0(t), \quad t \in [0,T], \tag{3b}$$
$$\theta(t,Z) = \theta_Z(t), \quad t \in [0,T]. \tag{3c}$$

In order to face this model, we will first resort to a classical TMoL approach and then we will integrate (1) and approximate its solution by means of the *shooting method*, which in light of boundary conditions (3) seems a natural strategy in this context.

2.1 Kirchhoff Transformation on Richards' Equation with Root-Water Uptake

Since, in practice, numerical methods for directly solving (1)–(3) with respect to θ could be highly unstable because of the nonlinear nature of Richards' equation (see [30]), it is reasonable to transform the θ-based form by a suitable change of variable, to obtain a new equation easier to handle using classical numerical schemes (see also the recent papers [31,32]). Resorting to matric flux potential allows to express (1) in a handier way, which comes from the classical Kirchhoff transformation used in heat equation; more precisely, we introduce the variable

$$\mu(\theta(t,z)) := \int_{\theta_r}^{\theta(t,z)} D(\tau)\,\mathrm{d}\tau, \tag{4}$$

which we plug into the boundary value problem (1)-(3).
Considering that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = D(\theta)$$

and replacing this in (1) it follows that

$$F(\mu)\frac{\partial \mu}{\partial t} = L\mu - R(\mu), \tag{5}$$

where

$$L := \frac{\partial^2}{\partial z^2} - G(\cdot)\frac{\partial}{\partial z} \tag{6}$$

is the *Kirchhoff operator*,

$$F(\mu) := \frac{1}{D(\theta)}, \quad G(\mu) := \frac{1}{D(\theta)}\frac{\mathrm{d}K}{\mathrm{d}\theta},$$

and $R(\mu)$ is identified with $R(\theta(\mu))$. Finally, we also get

$$\mu(0,z) = \mu^0(z), \quad z \in [0,Z], \tag{7a}$$
$$\mu(t,0) = \mu_0(t), \quad t \in [0,T], \tag{7b}$$
$$\mu(t,Z) = \mu_Z(t), \quad t \in [0,T]. \tag{7c}$$

Remark 1. If Gardner soil model is considered, then

$$K(h) = K_s e^{\alpha h},$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Let us stress that $\frac{\mathrm{d}K}{\mathrm{d}h} = \alpha K(h)$. Now, from (2), it follows that $D(\theta) = \frac{1}{\alpha}\frac{\mathrm{d}K}{\mathrm{d}\theta}$, and so

$$G(\mu) = \alpha.$$

Thus, within the Gardner model, the Kirchhoff operator $L := \frac{\partial^2}{\partial z^2} - \alpha\frac{\partial}{\partial z}$ is a *linear* elliptic differential operator.

2.2 Solving Richards' Equation Applying Shooting Method

For applying TMoL to (5)–(7), the time derivative operator in the left-hand side in (1) is approximated through a first order finite difference, choosing a time-step Δt and a time mesh $t_n = n\Delta T$, $n = 0, \ldots, N$, for $[0, T]$ such that $T = N\Delta t$. Let $\mu^0(z)$ $z \in [0, Z]$ be a known guess of the solution to (5) at time $t = 0$, we focus on the following second order ordinary differential equation in space

$$F(\mu^n)\frac{\mu^n - \mu^{n-1}}{\Delta t} = L\mu^n - R(\mu^n), \quad n = 1, \ldots, N \tag{8}$$

for $z \in [0, Z]$, with boundary conditions

$$\mu^n(0) = \mu_0(t^n), \quad \mu^n(Z) = \mu_Z(t^n), \quad n = 1, \ldots, N. \tag{9}$$

Rearranging (8) and exploiting (6), we obtain

$$\frac{\mathrm{d}^2\mu^n(z)}{\mathrm{d}z^2} = \frac{F(\mu^n(z))(\mu^n(z) - \mu^{n-1}(z))}{\Delta t} + R_n + \alpha\nu^n, \tag{10}$$

where $\mu^n(0)$ and $\mu^n(Z)$ are given.

Next we will prove that, under the given assumptions on the functions R and F, solution to the above problem (8) and (9) exists and is unique.

For sake of clarity, we state the argument in a more general setting. Let us assume we have a nonlinear second-order boundary value problem

$$y''(x) = f(x, y, y'), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta. \tag{11}$$

It is just an observation that (10) is of the form described by (11).
We have the following result.

Theorem 1 ([33]). *Assume that $f(x, y, z)$ is continuous on the region $R = \{(x, y, z) : a \leq x \leq b, y, z \in \mathbb{R}\}$ and that $\frac{\partial f}{\partial y} = f_y(x, y, z)$ and $\frac{\partial f}{\partial z} = f_z(x, y, z)$ are continuous on R; further, let α, $\beta \in \mathbb{R}$. If there exists a constant $M > 0$ for which f_y and f_z satisfy*

$$f_y(x, y, z) > 0, \text{ for all } (x, y, z) \in R, \tag{12a}$$

$$|f_z(x, y, z)| \leq M, \text{ for all } (x, y, z) \in R, \tag{12b}$$

then the boundary value problem

$$y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta$$

has a unique solution $y = y(x)$ *for* $a \leq x \leq b$.

In order to numerically solve (11) we consider, for each $k \in \mathbb{N}$, the following initial value problem

$$y''(x) = f(x, y, y'), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y'(a) = \alpha'_k, \tag{13}$$

where $\alpha'_k \in \mathbb{R}$. The idea is to select the sequence of initial value problems, depending on $\{\alpha'_k\}_{k\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} y(b, \alpha'_k) = y(b) = \beta, \tag{14}$$

where $y(x, \alpha'_k)$ denotes the solution to (13), for a given $k \in \mathbb{N}$, and $y(x)$ is the solution to (11). We select α'_0 for starting the limit process (14) solving

$$y''(x) = f(x, y, y'), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y'(a) = \alpha'_0.$$

Then, if $y(b, \alpha'_0)$ is not sufficiently close to β, we keep solving (13) with different values α'_1, α'_2 and so on until $y(b, \alpha'_k)$ is sufficiently close to β. Determining the sought parameter α'_k requires solving a zero-finding problem of the form

$$y(b, \alpha') - \beta = 0$$

with respect to α'. Under the assumptions of Theorem 1, last problem could be solved by using Newton's method, considering the sequence

$$\alpha'_k = \alpha'_{k-1} - \frac{y(b, \alpha'_{k-1}) - \beta}{\eta(b, \alpha'_{k-1})}, \tag{15}$$

where $\eta(x, \alpha')$ solves the following initial value problem

$$\eta''(x, \alpha') = \frac{\partial f}{\partial y}(x, y, y')\eta(x, \alpha') + \frac{\partial f}{\partial y'}(x, y, y')\eta'(x, \alpha'), \ \eta(a, \alpha') = 0, \ \eta'(a, \alpha') = 1,$$

for $a \leq x \leq b$.

In order to leverage (15) for solving (8)–(9), we need that

$$f(z, \mu^n, \nu^n) := \frac{F(\mu^n(z))(\mu^n(z) - \mu^{n-1}(z))}{\Delta t} + R_n + \alpha\nu^n$$

fulfills assumptions of Theorem 1. Easy computations provide

$$\frac{\partial f}{\partial \mu^n} = \frac{F'(\mu^n)(\mu^n - \mu^{n-1}) + F(\mu^n)}{\Delta t} + R'(\mu^n), \tag{16}$$

$$\frac{\partial f}{\partial \nu^n} = \alpha. \tag{17}$$

From Eqs. (16) and (17) it follows that (12b) is always satisfied; whereas, in order for (12a) to hold, it suffices to prove that $\mu F(\mu)$ is increasing, since in common scenarios root-water uptake term R is usually increasing as a function of μ, so that $R'(\mu) > 0$. This is a consequence of the following.

Proposition 1. *Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ be differentiable over I and let us assume that, for all $x \in I$, $f(x) > 0$, $f'(x) < 0$ and $f''(x) > 0$. Then the function $x \mapsto xf(x)$ is strictly increasing for all $x \in I$.*

Proof. Let $x_0, x_1 \in I$ and $x_0 < x_1$. By contradiction, assume $x_0 f(x_0) \geq x_1 f(x_1)$. Thus

$$x_0 f(x_0) \geq (x_1 - x_0) f(x_1) + x_0 f(x_1).$$

Rearranging term yields

$$-x_0(f(x_1) - f(x_0)) \geq (x_1 - x_0) f(x_1),$$

and using the fact that

$$f(x_1) - f(x_0) \geq f'(x_0)(x_1 - x_0)$$

it follows that

$$-x_0 f'(x_0) \geq f(x_1),$$

since $x_1 - x_0 > 0$. But $f(x_1) > 0$, giving a contradiction.

We observe now that, in Gardner model, F satisfies conditions of Proposition 1; moreover, since $\mu^{n-1}(z)$ is always positive – on the account of (4), then $F(\mu^n)(\mu^n - \mu^{n-1})$ is increasing as a function of μ^n, since so is $F(\mu^n)\mu^n$. Hence, Theorem 1 guarantees existence and uniqueness of solution to (10).

The idea now is to express the boundary value problem (8) and (9) as a first order differential equation endowed with suitable initial conditions, so that this new differential problem yields the same solution as the first one: this is the core idea of *shooting method* for solving boundary value problems.

Similarly to [15], we notice that Eq. (8) can be written as a first order system of differential equations of the form

$$\frac{\mathrm{d}\mu^n}{\mathrm{d}z} = \nu^n, \tag{18a}$$

$$\frac{\mathrm{d}\nu^n}{\mathrm{d}z} = F(\mu^n)\frac{\mu^n - \mu^{n-1}}{\Delta t} + G(\mu^n)\nu^n + R(\mu^n), \tag{18b}$$

which we endow with the following *initial conditions*

$$\mu^n(0) = \mu_0(t^n), \quad \nu^n(0) = \hat{\nu}, \tag{19}$$

$\hat{\nu}$ being computed by standard numerical methods (see, for instance, [34]) in such a way that second condition in (9) is met, that is

$$\mu^n(Z) = \mu_Z(t_n). \tag{20}$$

3 Numerical Simulations

Example 1. From [35] a loamy sand is considered with the following hydraulic parameters:

$$\theta_r = 0.057, \quad \theta_S = 0.41, \quad K_S = 350.2\,\text{cm/d}. \tag{21}$$

For this example, we consider a simulation time of $T = 3$ h and a spatial domain of $Z = 20$ cm.

In order to apply our numerical methods to (8) for this specific soil, it is advisable to first scale quantities and operators so to simplify the equations, as in [22]; in particular, we set

$$\Theta := \frac{\theta - \theta_r}{\theta_S - \theta_r}. \tag{22}$$

For exemplification purposes, Dirichlet boundary conditions will only be considered in the numerical computations displayed in Fig. 1.

Moreover, we will restrict ourselves to the setting of non-classical symmetry classification: that is, soil-water diffusivity and root-water uptake satisfy the condition

$$-R(\mu) = A\frac{\mu}{D(\mu)} + \kappa\mu. \tag{23}$$

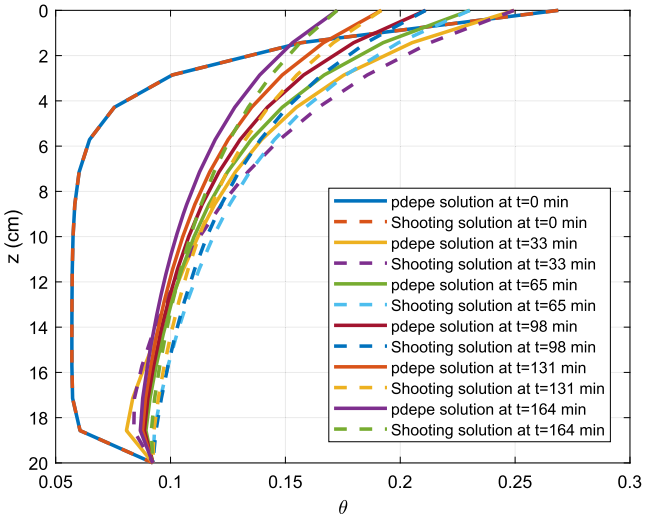


Fig. 1. Numerical simulations for the loamy sand with parameters (21), obtained by shooting method, compared with the solution arising from MATLAB `pdepe` tool.

Specifically, as in [22], we will consider the functions[1]:

$$D(\mu) := ce^{\mu},$$
$$R(\mu) := \mu(1 - e^{-\mu}),$$

with $c = 1 - e^{-1}$ and $k = -1$.

Further, dealing with such re-scaled parameters, for all $t \geq 0$ we consider a time-varying top boundary condition at $z = 0$ given by

$$\Theta(t, 0) = \frac{2T - t}{2T} 0.6, \tag{24}$$

meaning that $\theta(0, 0) = 0.2688$; and a constant bottom condition at $z = Z$ given by

$$\Theta(t, Z) = 0.1, \tag{25}$$

implying that $\theta(0, Z) = 0.0923$.

Let us point out that, after determining the normalized solution $\Theta(t, z)$ (either by shooting or `pdepe` method) we go back to original water content by computing

$$\theta(t, z) = (1 - \Theta(t, z))\theta_r + \Theta(t, z)\theta_S.$$

Within this specific setting, we use as step-sizes $\Delta t = 8.8974$ min, and $\Delta z \approx 1.3$ cm when comparing `pdepe` and the shooting method: we point out that the shooting method runs using a temporal step-size of one order less than `pdepe`, while retaining similar performances (see Figs. 1 and 2).

In Fig. 1 we build numerical solutions of (18) with parameters as in (21) and initial condition as suggested in [22] and according to the aforementioned hydraulic functions and parameters, for comparing shooting solution to `pdepe` solution. The initial condition at time $t = 0$ is a linear combination of exponential functions of the form

$$\Theta^0(z) := c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z}, \tag{26}$$

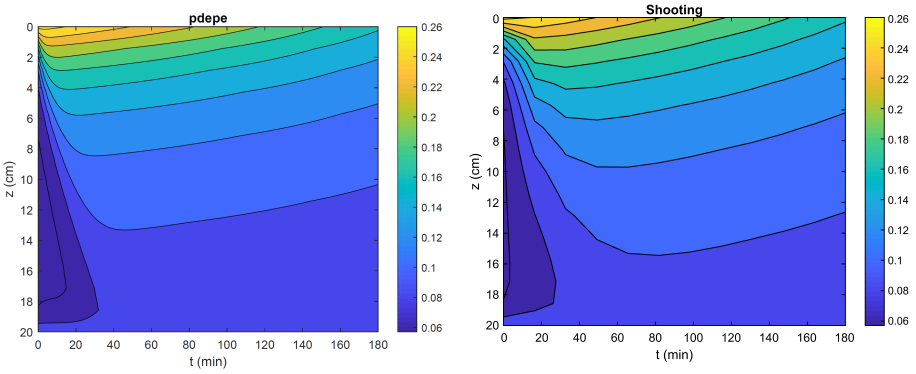
such that $\Theta^0(0) = 0.6$, $\Theta^0(Z) = 0.1$. Thus, after straightforward computations, we get

$$c_1 = \frac{0.1e^{-\lambda_2 Z} - 0.6}{e^{\lambda_2 Z} - e^{\lambda_1 Z}},$$
$$c_2 = \frac{0.1 - 0.6e^{-\lambda_1 Z}}{e^{\lambda_2 Z} - e^{\lambda_1 Z}},$$

with $\lambda_{1,2}$ chosen arbitrarily but with different sign, so to make (26) retain a physical meaning.

Figure 2 summarizes the behavior of different methods in this example.

[1] We highlight that the uptake function R should be used as reported here, instead as the one originally proposed in [22], in order for (23) to hold; the function D stays the same.



(a) Pdepe solution: Example 1.

(b) Shooting method solution: Example 1.

Fig. 2. Dynamics of water content over time for the aforementioned problem, referring to a loamy sand in 1, with an uptake function as in [22]. The figures refer, respectively, to the MATLAB `pdepe` solution and the shooting method, described in Sect. 2.

Example 2. Here, we consider a silty loam Swift current wheat site. We fix $Z = 70$ cm and $T = 3$ days, while the following hydraulic parameters are as follows:

$$\theta_r = 0, \quad \theta_S = 0.48, \quad K_S = 100.1\,\text{cm/d}$$

We select boundary conditions as

$$\theta(t,0) = 0.3610,\ \theta(t,Z) = 0.2051, \quad t \in [0,T],$$

while initial condition is chosen to be

$$\theta(0,z) = \theta(0,0) + \frac{\theta(0,Z) - \theta(0,0)}{Z} z.$$

In this case example, we consider a different root uptake model that in Example 1, which would not require any re-scaling in order to be fruitfully handled. More specifically, we adopt a Feddes-type stress function with the following form:

$$R = R_{\max} \alpha\,(h)\,, \tag{27}$$

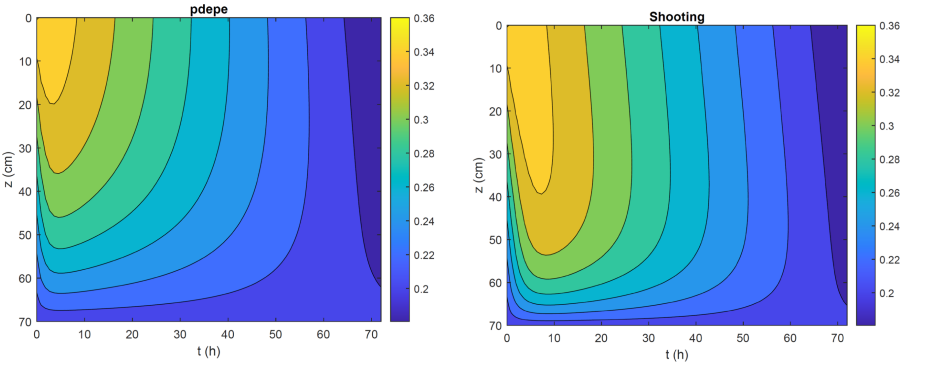
where $R_{\max}$ is the so-called *maximum uptake*, representing the potential root water uptake (see [21]), and the function $\alpha(h)$ is defined, for $h_4 < h_3 < h_2 < h_1 < 0$, as

$$\alpha(h) := \begin{cases} 0, & \text{if } h_1 \le h \le 0 \text{ or } h \le h_4, \\ \frac{h-h_1}{h_2-h_1}, & \text{if } h_2 < h < h_1, \\ 1, & \text{if } h_3 \le h \le h_2, \\ \frac{h-h_4}{h_3-h_4}, & \text{if } h_4 < h < h_3, \end{cases} \tag{28}$$

and represents the stress response function.

Such a function is piecewise linear on the interval $[h_4, 0]$, so that $\alpha(h)$ is just continuous with finitely many jumps in its derivative. Let us stress that results in Sect. 2 still applies, and that - from a numerical point of view - a wise selection of the discretized grid allows to overlook such points, where the numerical integration may become stiff: this example shows, actually, that the shooting method is also robust with respect to possibly mildly nonsmooth uptake models.

Figure 3 depicts the solutions computed through MATLAB `pdepe` and the Shooting method (18) for this specific soil setting, showing their similar behaviors.



(a) Pdepe solution: Example 2.

(b) Shooting method solution: Example 2.

Fig. 3. Dynamics of water content over time for the Example in 2. The soil is a silty loam, and the uptake function is of Feddes type, cropped with wheat. The figures refer, respectively, to the MATLAB `pdepe` solution and the shooting method, described in Sect. 2.

Example 3. Let us consider the same soil parameters as in Example 2, with again $T = 3$ days and $Z = 70$ cm. Same initial and boundary conditions as in Example 2 are selected.

For this example a Li uptake function (see [36]) is now modeling $\alpha(h)$ in (27). More specifically, we select

$$\alpha(h) := \begin{cases} \frac{1}{1+e^{12.25\left(0.504-\frac{\theta(h_0)-\theta(h)}{\theta(h_0)-\theta(h_{fc})}\right)}}, & h_{fc} < h < 0, \\ \min\left(1, \frac{(H(h)\tau(h))^{0.5}}{T_p/T_m}\right), & h_{pwp} \le h \le h_{fc}, \\ 0, & h < h_{pwp}, \end{cases}$$

being

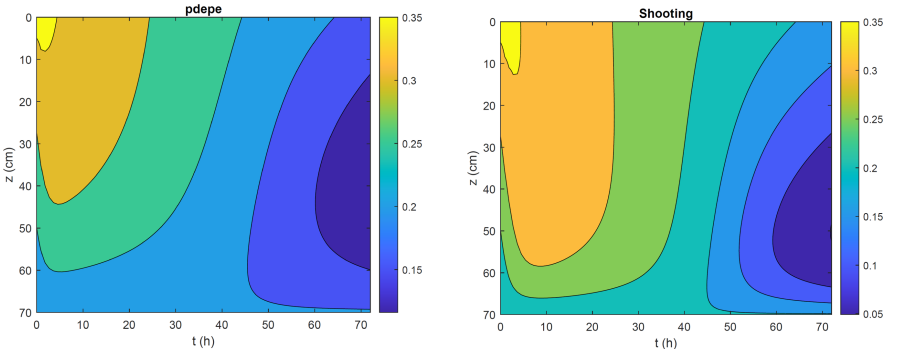
$$H(h) := \frac{h - h_{pwp}}{h_{fc} - h_{pwp}}, \quad \tau(h) := \frac{\theta(h) - \theta(h_{pwp})}{\theta(h_{fc}) - \theta(h_{pwp})}, \ h_{pwp} < h_{fc} < 0.$$

Again, such an uptake model $\alpha(h)$ is only continuous over the interval $[h_{pwp}, 0]$, but nonetheless shooting method performs well and provides convergence, as expected from Sect. 2.

Here, as in [36], the values of field capacity and permanent wilting point are, respectively, $h_{fc} = -3$ m, $h_{pwp} = -160$ m, the maximum potential transpiration rate $T_m = 7.06 \cdot 10^{-8}$ m/s and the potential transpiration rate $T_p = 7.407 \cdot 10^{-8}$ m/s; finally, $\theta(h)$ is defined, as in [32],

$$\theta(h) = \begin{cases} \theta_r + (\theta_S - \theta_r)e^{\lambda h}, & \text{for } h \leq 0, \\ 0, & \text{for } h > 0. \end{cases} \tag{29}$$

A comparison between `pdepe` and the shooting method is depicted in Fig. 4. As it can be seen, the shooting method keeps retaining a solution shape similar to `pdepe` but, as shown in [15], it can manage more general and specific uptake models and boundary conditions than `pdepe`.



(a) Pdepe solution: Example 3.

(b) Shooting method solution: Example 3.

Fig. 4. Water movement over time for the problem in Example 3, still regarding a silty loam soil. The figures depict, respectively, MATLAB `pdepe` and the shooting method solutions.

4 Conclusions and Future Works

In this paper we propose the theoretical background for the shooting method applied to Richards' equation with a sink term as proposed in [15]. We show convergence results and exemplify on several soils with different uptake models. A new approach, recently proposed in [37], rewrites (1) as an integral equation and solves it by standard quadrature schemes. We anticipate to tackle such an integral equation in presence of layered soils, and integrate it using a peridynamics formulation, leveraging techniques developed in [38–40]). Modeling memory terms of root water uptake (e.g. [41–43]) can also be faced in future mathematical works, maybe also considering variations due to heat transfer [44].

Acknowledgments. The first author has been funded by *REFIN* Project, grant number 812E4967. The second author acknowledges the partial support of the project RIUBSAL (grant number 030_DIR_2020_00178), funded by Regione Puglia through the P.S.R. Puglia 2014/2020 - Misura 16 – Cooperazione - Sottomisura 16.2 call.

References

1. Maggi, S.: Estimating water retention characteristic parameters using differential evolution. Comput. Geotech. **86**, 163–172 (2017). https://doi.org/10.1016/j.compgeo.2016.12.025
2. Friedman, S.P., Communar, G., Gamliel, A.: DIDAS - user-friendly software package for assisting drip irrigation design and scheduling. Comput. Electron. Agricult. **120**, 36–52 (2016). https://doi.org/10.1016/j.compag.2015.11.007
3. Brunetti, G., Šimůnek, J., Bautista, E.: A hybrid finite volume-finite element model for the numerical analysis of furrow irrigation and fertigation. Comput. Electron. Agricult. **150**, 312–327 (2018). https://doi.org/10.1016/j.compag.2018.05.013
4. Arbat, G., Puig-Bargués, J., Duran-Ros, M., Barragán, J., de Cartagena, F.R.: Drip-Irriwater: computer software to simulate soil wetting patterns under surface drip irrigation. Comput. Electron. Agricult. **98**, 183–192 (2013). https://doi.org/10.1016/j.compag.2013.08.009
5. Berardi, M., Difonzo, F., Notarnicola, F., Vurro, M.: A transversal method of lines for the numerical modeling of vertical infiltration into the vadose zone. Appl. Numer. Math. **135**, 264–275 (2019). https://doi.org/10.1016/j.apnum.2018.08.013
6. Berardi, M., Difonzo, F.V.: Strong solutions for Richards' equation with Cauchy conditions and constant pressure gradient. Environ. Fluid Mech. **20**(1), 165–174 (2019). https://doi.org/10.1007/s10652-019-09705-w
7. Casulli, V.: A coupled surface-subsurface model for hydrostatic flows under saturated and variably saturated conditions. Int. J. Numer. Meth. Fluids **85**(8), 449–464 (2017). https://doi.org/10.1002/fld.4389
8. Nordbotten, J.M., Boon, W.M., Fumagalli, A., Keilegavlen, E.: Unified approach to discretization of flow in fractured porous media. Comput. Geosci. **23**(2), 225–237 (2018). https://doi.org/10.1007/s10596-018-9778-9
9. Berre, I., et al.: Verification benchmarks for single-phase flow in three-dimensional fractured porous media. Adv. Water Resour. **147**, 103759 (2021). https://doi.org/10.1016/j.advwatres.2020.103759
10. Berardi, M.: Rosenbrock-type methods applied to discontinuous differential systems. Math. Comput. Simul. **95**, 229–243 (2014). https://doi.org/10.1016/j.matcom.2013.05.006
11. Lopez, L., Maset, S.: Time-transformations for the event location in discontinuous ODEs. Math. Comp. **87**(3), 2321–2341 (2018). https://doi.org/10.1090/mcom/3305
12. Del Buono, N., Elia, C., Lopez, L.: On the equivalence between the sigmoidal approach and Utkin's approach for piecewise-linear models of gene regulatory. SIAM J. Appl. Dyn. Syst. **13**(3), 1270–1292 (2014). https://doi.org/10.1137/130950483
13. Berardi, M., Difonzo, F., Vurro, M., Lopez, L.: The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities. Adv. Water Resour. **115**, 264–272 (2018). https://doi.org/10.1016/j.advwatres.2017.09.027
14. Berardi, M., Difonzo, F., Lopez, L.: A mixed MoL-TMoL for the numerical solution of the 2D Richards' equation in layered soils. Comput. Math. Appl. **79**(7), 1990–2001 (2019). https://doi.org/10.1016/j.camwa.2019.07.026

15. Difonzo, F.V., Masciopinto, C., Vurro, M., Berardi, M.: *Shooting* the numerical solution of moisture flow equation with root water uptake models: a Python tool. Water Resour. Manage. **35**(8), 2553–2567 (2021). https://doi.org/10.1007/s11269-021-02850-2
16. Camporese, M., Daly, E., Paniconi, C.: Catchment-scale Richards equation-based modeling of evapotranspiration via boundary condition switching and root water uptake schemes. Water Resour. Res. **51**(7), 5756–5771 (2015). https://doi.org/10.1002/2015WR017139
17. Manoli, G., Bonetti, S., Scudiero, E., Morari, F., Putti, M., Teatini, P.: Modeling soil-plant dynamics: assessing simulation accuracy by comparison with spatially distributed crop yield measurements. Vadose Zone J. **14**(2) (2015), vzj2015-05. https://doi.org/10.2136/vzj2015.05.0069
18. Roose, T., Fowler, A.: A model for water uptake by plant roots. J. Theor. Biol. **228**(2), 155–171 (2004). https://doi.org/10.1016/j.jtbi.2003.12.012
19. Couvreur, V., Vanderborght, J., Javaux, M.: A simple three-dimensional macroscopic root water uptake model based on the hydraulic architecture approach. Hydrol. Earth Syst. Sci. **16**(8), 2957–2971 (2012). https://doi.org/10.5194/hess-16-2957-2012
20. Albrieu, J.L.B., Reginato, J.C., Tarzia, D.A.: Modeling water uptake by a root system growing in a fixed soil volume. Appl. Math. Model. **39**(12), 3434–3447 (2015). https://doi.org/10.1016/j.apm.2014.11.042
21. Simunek, J., Hopmans, J.W.: Modeling compensated root water and nutrient uptake. Ecol. Model. **220**(4), 505–521 (2009). https://doi.org/10.1016/j.ecolmodel.2008.11.004
22. Broadbridge, P., Daly, E., Goard, J.: Exact solutions of the Richards equation with nonlinear plant-root extraction. Water Resour. Res. **53**(11), 9679–9691 (2017). https://doi.org/10.1002/2017WR021097
23. Mathur, S., Rao, S.: Modeling water uptake by plant roots. J. Irrig. Drain. Eng. **125**(3), 159–165 (1999). https://doi.org/10.1061/(ASCE)0733-9437(1999)125:3(159)
24. D'Abbicco, M., Buono, N.D., Gena, P., Berardi, M., Calamita, G., Lopez, L.: A model for the hepatic glucose metabolism based on hill and step functions. J. Comput. Appl. Math. **292**, 746–759 (2016). https://doi.org/10.1016/j.cam.2015.01.036
25. Feddes, R.A., Kowalik, P., Kolinska-Malinka, K., Zaradny, H.: Simulation of field water uptake by plants using a soil water dependent root extraction function. J. Hydrol. **31**(1), 13–26 (1976). https://doi.org/10.1016/0022-1694(76)90017-2
26. Berardi, M., Girardi, G., D'Abbico, M., Vurro, M.: Optimizing water consumption in Richards' equation framework with step-wise root water uptake: a simplified model (2021, submitted)
27. Jarvis, N.: A simple empirical model of root water uptake. J. Hydrol. **107**(1), 57–72 (1989). https://doi.org/10.1016/0022-1694(89)90050-4
28. Babazadeh, H., Tabrizi, M., Homaee, M.: Assessing and modifying macroscopic root water extraction basil (ocimum basilicum) models under simultaneous water and salinity stresses. Soil Sci. Soc. Am. J. **81**, 10–19 (2017). https://doi.org/10.2136/sssaj2016.07.0217
29. Coppola, A., Chaali, N., Dragonetti, G., Lamaddalena, N., Comegna, A.: Root uptake under non-uniform root-zone salinity. Ecohydrology **8**(7), 1363–1379 (2015). https://doi.org/10.1002/eco.1594

30. Bergamaschi, L., Putti, M.: Mixed finite elements and Newton-type linearizations for the solution of Richards' equation. Int. J. Numer. Meth. Eng. **45**, 1025–1046 (1999). https://doi.org/10.1002/(SICI)1097-0207(19990720)45:8⟨1025::AID-NME615⟩3.0.CO;2-G
31. Li, N., Yue, X., Ren, L.: Numerical homogenization of the Richards equation for unsaturated water flow through heterogeneous soils. Water Resour. Res. **52**(11), 8500–8525 (2016). https://doi.org/10.1002/2015WR018508
32. Suk, H., Park, E.: Numerical solution of the Kirchhoff-transformed Richards equation for simulating variably saturated flow in heterogeneous layered porous media. J. Hydrol. **579**, 124213 (2019). https://doi.org/10.1016/j.jhydrol.2019.124213
33. Mathews, J.H., Fink, K.K.: Numerical Methods Using MATLAB, 4th edn. Pearson, New York (2004)
34. D'Ambrosio, R., Jackiewicz, Z.: Construction and implementation of highly stable two-step continuous methods for stiff differential systems. Math. Comput. Simul. **81**(9), 1707–1728 (2011). https://doi.org/10.1016/j.matcom.2011.01.005
35. Carsel, R.F., Parrish, R.S.: Developing joint probability distributions of soil water retention characteristics. Water Resour. Res. **24**(5), 755–769 (1988). https://doi.org/10.1029/WR024i005p00755
36. Li, K.Y., De Jong, R., Coe, M.T., Ramankutty, N.: Root-water-uptake based upon a new water stress reduction and an asymptotic root distribution function. Earth Interact. **10**(14), 1–22 (2006)
37. Berardi, M., Difonzo, F.: A quadrature-based scheme for numerical solutions to Kirchhoff transformed Richards' equation (2021, submitted)
38. Lopez, L., Pellegrino, S.F.: A spectral method with volume penalization for a nonlinear peridynamic model. Int. J. Numer. Meth. Eng. **122**(3), 707–725 (2020). https://doi.org/10.1002/nme.6555
39. Lopez, L., Pellegrino, S.F.: A space-time discretization of a nonlinear peridynamic model on a 2D lamina (2021)
40. Lopez, L., Pellegrino, S.F., Coclite, G., Maddalena, F., Fanizzi, A.: Numerical methods for the nonlocal wave equation of the peridynamics. Appl. Numer. Math. **155**, 119–139 (2018). https://doi.org/10.1016/j.apnum.2018.11007
41. Carminati, A.: A model of root water uptake coupled with rhizosphere dynamics. Vadose Zone J. **11**, vzj2011-0106 (2012). https://doi.org/10.2136/vzj2011.0106
42. Kroener, E.: How mucilage affects soil hydraulic dynamics. J. Plant Nutr. Soil Sci. **184**, 20–24 (2021). https://doi.org/10.1002/jpln.202000545
43. Wu, X., Zuo, Q., Shi, J., Wang, L., Xue, X., Ben-Gal, A.: Introducing water stress hysteresis to the Feddes empirical macroscopic root water uptake model. Agricult. Water Manage. **240**, 106293 (2020). https://doi.org/10.1016/j.agwat.2020.106293
44. Tubini, N., Gruber, S., Rigon, R.: A method for solving heat transfer with phase change in ice or soil that allows for large time steps while guaranteeing energy conservation. Cryosphere Discuss. **2020**, 1–41 (2020). https://doi.org/10.5194/tc-2020-293