

Optimal ϑ-Methods for Mean-Square Dissipative Stochastic Differential Equations

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Abstract. In this paper, we address our investigation to the numerical integration of nonlinear stochastic differential equations exhibiting a mean-square contractive character along the exact dynamics. We specifically focus on the conservation of this qualitative feature along the discretized dynamics originated by applying stochastic ϑ-methods. Retaining the mean-square contractivity under time discretization is translated into a proper stepsize restriction. Here we analyze the choice of the optimal parameter ϑ making this restriction less demanding and, at the same time, maximizing the stability interval. A numerical evidence is provided to confirm our theoretical results.

Keywords: Stochastic differential equations · Stochastic ϑ-methods · Mean-square dissipativity · Mean-square contractivity

1 Introduction

We consider a system of stochastic differential equations (SDE) of Itô type assuming the following differential form

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t), \quad t \in [0,T], \\ X(0) = X_0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $W(t)$ is a m-dimensional Wiener process. For theoretical results on the existence and uniqueness of solutions to (1) we refer, for instance, to the classical monograph [25]. Moreover, in the sequel, we assume that the diffusive term in (1) is commutative.

We focus our attention on providing a nonlinear stability analysis of the following stochastic ϑ-methods for (1) that, with reference to the discretized domain

$$\mathcal{I}_{\Delta t} = \{t_n = n\Delta T,\ n = 0, 1, \dots, N,\ N = T/\Delta t\},$$

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assume the following forms:

$$X_{n+1} = X_n + (1-\vartheta)\Delta t f(X_n) + \vartheta \Delta t f(X_{n+1}) + g(X_n)\Delta W_n, \tag{2}$$

$$X_{n+1} = X_n + (1-\vartheta)\Delta t f(X_n) + \vartheta \Delta t f(X_{n+1}) + \sum_{j=1}^{m} g^j(X_n)\Delta W_n^j \tag{3}$$
$$+ \frac{1}{2}\sum_{j=1}^{m} L^j g^j(X_n)((\Delta W_n^j)^2 - \Delta t) + \frac{1}{2}\sum_{\substack{j_1,j_2=1 \\ j_1 \neq j_2}}^{m} L^{j_1} g^{j_2}(X_n)\Delta W_n^{j_1}\Delta W_n^{j_2},$$

where $\vartheta \in [0,1]$. X_n is the approximate value for $X(t_n)$ and ΔW_n is the discretized Wiener increment, distributed as a gaussian random variable with zero mean and variance Δt. The operator L^j is defined as

$$L^j = \sum_{k=1}^{n} g^{k,j}\frac{\partial}{\partial x^k}, \quad j = 1, ..., m,$$

where $g^j(X_n)$ is the $j-$th column of the matrix $g(X_n)$ and ΔW_n^j the $j-$th element of vector ΔW_n. We refer to (2) as ϑ-*Maruyama* methods and to (3) as ϑ-*Milstein* methods in their componentwise form. We note that, if $m = 1$, (3) reduces to the form

$$X_{n+1} = X_n + (1-\vartheta)\Delta t f(X_n) + \vartheta \Delta t f(X_{n+1}) + g(X_n)\Delta W_n + \frac{1}{2}g(X_n)g'(X_n)(\Delta W_n^2 - \Delta t).$$

Convergence and linear stability analysis for stochastic ϑ-methods (2)–(3) has been investigated in [3,20] and reference therein. Successively, a nonlinear stability analysis has been performed in [12], as well as their conservation properties when applied to linear and nonlinear stochastic oscillators [6,16]. The idea of ϑ-methods has also been extended to stochastic Volterra integral equations, in recent contributions [9,10].

1.1 Mean-Square Contractivity

Let us focus on the following relevant issue for nonlinear SDEs [22].

Theorem 1. *For a given nonlinear SDE (1), let us assume the following properties for the drift f and the diffusion g, by denoting with $|\cdot|$ both the Euclidean norm in $\mathbb{R}^n$ and the trace (or Frobenius) norm in $\mathbb{R}^{n\times m}$:*

(i) $f, g \in \mathcal{C}^1(\mathbb{R}^n)$;
(ii) *f satisfies a one-sided Lipschitz condition, i.e. there exists $\mu \in \mathbb{R}$ such that*

$$< x-y, f(x)-f(y) > \leq \mu |x-y|^2, \quad \forall x, y \in \mathbb{R}^n; \tag{4}$$

(iii) *g is a globally Lipschitz function, i.e. there exists $L > 0$ such that*

$$|g(x)-g(y)|^2 \leq L|x-y|^2 \quad \forall x, y \in \mathbb{R}^n. \tag{5}$$

Then, any two solutions $X(t)$ and $Y(t)$ of (1), with $\mathbb{E}\,|X_0|^2 < \infty$ and $\mathbb{E}\,|Y_0|^2 < \infty$, satisfy

$$\mathbb{E}\,|X(t) - Y(t)|^2 \leq \mathbb{E}\,|X_0 - Y_0|^2\, e^{\alpha t}, \tag{6}$$

where $\alpha = 2\mu + L$.

We call the inequality (6) *exponential mean-square stability inequality* for (1). If $\alpha < 0$ in (6), then an exponential decay of the mean-square deviation between two solutions of a given SDE (1) occurs. We provide the following definition that appears as the stochastic counterpart of a similar property for deterministic ordinary differentia equations (see, for instance, [18] and references therein).

Definition 1. *A nonlinear SDE (1) whose solutions satisfy the exponential stability inequality (6) with $\alpha < 0$ is said to be exponential mean-square dissipative and to generate exponential mean-square contractive solutions.*

If $g = 0$ in (1), Definition 1 overlaps with the deterministic analog provided by $\mu = 0$ guaranteeing a dissipative behavior of the deterministic solutions to the corresponding ordinary differential problems. Their discretizations have led to the well-known notion of G-stability of numerical methods, introduced by G. Dahlquist in [11]. In the existing literature, there are several papers in which deterministic and stochastic problems with one-sided Lipschitz constants appear. In particular, they find huge application in finance and economics, neural networks, population dynamics, systems synchronization, stochastic differential equations with jumps, dynamical systems, delay ordinary (and stochastic) differential equations (see [4,8,24,26,28,30–32,36] and references therein).

The results presented in [12] shows that it is possible to retain the mean-square contractivity under time discretization through a proper stepsize restriction. A rigorous discussion of the optimal choice of the parameter ϑ, however, is missing in [12] and filling this gap in represents the goal of this contribution. Indeed, we discuss the choice of the optimal parameter ϑ making the aforementioned stepsize restrictions less demanding and, at the same time, maximizing the stability interval. As observed in [12], exponential mean-square contractivity is certainly a relevant property to be inherited also by the discretized problem, since it ensures a long-term damping of the error along the numerical solutions. Specifically, the investigation led to sharp restrictions on the stepsize employed in the numerical discretization to guarantee the conservation of exponential contractive behavior along the numerical dynamics.

It is worth observing that our research is in line with the idea of structure-preserving numerical integration; contributions in this direction and regarding stochastic numerics are given, for instance, in [1,2,5,7,13,27,29,33,35] and references therein.

The paper is organized as follows: in Sect. 2 we report results on linear and nonlinear stability of stochastic ϑ-methods (2)–(3), while in Sect. 3 we focus on the analysis of the optimal choice for the parameter ϑ guaranteeing the least demanding stepsize restriction to retain mean-square contractivity along the exact dynamics and maximal linear stability region; some concluding remarks are object of Sect. 4.

2 Stability Analysis of Stochastic ϑ-Methods

In this section, we summarize the main results about linear and nonlinear stability properties of stochastic ϑ-methods (2)–(3), that provide the starting point for our investigation. This section is totally inspired by [12,19–21].

2.1 Linear Stability

Linear stability analysis relies on considering the following scalar linear test equation

$$\begin{cases} dX(t) = \lambda X(t)dt + \sigma X(t)dW(t), \quad t \in [0,T], \\ X(0) = X_0, \end{cases} \tag{7}$$

where $\lambda, \sigma \in \mathbb{C}$. We recall the following definition [19–21].

Definition 2. *The solution $X(t)$ to the linear SDE (7) is mean-square stable if*

$$\lim_{t\to\infty} \mathbb{E}\,|X(t)|^2 = 0. \tag{8}$$

It can been shown that, in order to attain (8), the coefficients of (7) have to satisfy

$$\mathrm{Re}(\lambda) + \frac{1}{2}|\sigma|^2 < 0. \tag{9}$$

Then, it looks natural to provide a numerical counterpart of mean-square stability also under time discretization, as follows.

Definition 3. *A numerical solution X_n to linear SDE (7) is said to be mean-square stable if*

$$\lim_{n\to\infty} \mathbb{E}\,|X_n|^2 = 0. \tag{10}$$

It has been proved in [19,20] that stochastic ϑ-Maruyama methods (2) are mean-square stable if and only if

$$\frac{|1+(1-\vartheta)\Delta t\lambda|^2 + \Delta t\,|\sigma|^2}{|1-\vartheta\Delta t\lambda|^2} < 1, \tag{11}$$

while the stochastic ϑ-Milstein satisfies (10) if and only if [12]

$$\left|\beta^2 + \frac{\beta\sigma^2\Delta t}{1-\vartheta\Delta t\lambda} + \frac{\sigma^2\Delta t + \frac{3}{4}\sigma^4\Delta t^2}{(1-\vartheta\Delta t\lambda)^2}\right| < 1, \tag{12}$$

with

$$\beta = \frac{1+(1-\vartheta)\Delta t\lambda - \frac{1}{2}\sigma^2\Delta t}{1-\vartheta\Delta t\lambda}. \tag{13}$$

2.2 Nonlinear Stability

Let us now recall some of the tools useful to analyze nonlinear stability of stochastic ϑ-methods (2)–(3), i.e., we aim to retain the same behaviour described by the exponential mean-square stability inequality (6). As regards stochastic ϑ-Maruyama methods, the following theorem holds true [12].

Theorem 2. *Under the assumptions (i)–(iii) given in Theorem 1, any two numerical solutions X_n and Y_n, $n \geq 0$, computed by applying the ϑ-Maruyama method (2) to (1) with initial values such that $\mathbb{E}\,|X_0|^2 < \infty$ and $\mathbb{E}\,|Y_0|^2 < \infty$, satisfy the inequality*

$$\mathbb{E}\,|X_n - Y_n|^2 \leq \mathbb{E}\,|X_0 - Y_0|^2\, e^{\nu(\vartheta,\Delta t)t_n}, \tag{14}$$

where

$$\nu(\vartheta,\Delta t) = \frac{1}{\Delta t}\ln\beta(\vartheta,\Delta t) \tag{15}$$

and

$$\beta(\vartheta,\Delta t) = 1 + \frac{\alpha + (1-\vartheta)^2 M\Delta t}{1 - 2\vartheta\mu\Delta t}\Delta t, \tag{16}$$

with

$$M = \sup_{t\in[0,T]} \mathbb{E}|f'(X(t))|^2. \tag{17}$$

We observe that (14) gives the numerical counterpart of the exponential inequality (6) stated for the SDE (1). A negative value of ν gives the exponential mean-square decay of the gap between two numerical solutions computed by (2). Moreover, it has been proved that the numerical exponent ν approaches the exact one α as $\Delta t \to 0$, as stated by the following result [12].

Theorem 3. *Under the same assumptions of Theorem 2, for any fixed value of $\vartheta \in [0,1]$, we have*

$$|\nu(\vartheta,\Delta t) - \alpha| = \mathcal{O}(\Delta t). \tag{18}$$

Analogous results have also been established for stochastic ϑ-Milstein schemes (3) in [12], as provided by the following theorems.

Theorem 4. *Under the assumptions (i)–(iii) given in Theorem 1, any two numerical solutions X_n and Y_n, $n \geq 0$, computed by applying the ϑ-Milstein method (3) to (1) with initial values such that $\mathbb{E}\,|X_0|^2 < \infty$ and $\mathbb{E}\,|Y_0|^2 < \infty$, satisfy the inequality*

$$\mathbb{E}\,|X_n - Y_n|^2 \leq \mathbb{E}\,|X_0 - Y_0|^2\, e^{\varepsilon(\vartheta,\Delta t)t_n}, \tag{19}$$

where

$$\varepsilon(\vartheta,\Delta t) = \frac{1}{\Delta t}\ln\gamma(\vartheta,\Delta t) \tag{20}$$

and

$$\gamma(\vartheta,\Delta t) = \beta(\vartheta,\Delta t) + \frac{3\widetilde{M}\Delta t^2}{4(1 - 2\vartheta\mu\Delta t)}, \tag{21}$$

with $\widetilde{M}$ *defined as*

$$\widetilde{M} = \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \widetilde{M}_{i,j}^{k,l}, \tag{22}$$

where

$$\widetilde{M}_{i,j}^{k,l} = \sup_{t\in[0,T]} \frac{\mathbb{E}\left(h_{i,j}^{k,l}(X(t),Y(t))\right)}{\mathbb{E}|X(t)-Y(t)|^2},$$

being

$$h_{i,j}^{k,l}(X(t),Y(t)) = < g^{k,i}(X(t))\frac{\partial}{\partial x^k}g^j(X(t)) - g^{k,i}(Y(t))\frac{\partial}{\partial y^k}g^j(Y(t)),$$
$$g^{l,i}(X(t))\frac{\partial}{\partial x^l}g^j(X(t)) - g^{l,i}(Y(t))\frac{\partial}{\partial y^l}g^j(Y(t)) >,$$

$i,j = 1,\dots,m,\ k,l = 1,\dots,n.$

Theorem 5. *Under the same assumptions of Theorem 4, for any fixed value of* $\vartheta \in [0,1]$*, we have*

$$|\varepsilon(\vartheta,\Delta t) - \alpha| = \mathcal{O}(\Delta t). \tag{23}$$

Clearly, inequality (19) reveals a mean-square contractive behavior along the numerical solutions computed by (3), under the condition $\varepsilon < 0$.

Theorems 2 and 4 are the needed ingredient to provide the numerical counterpart of the condition $\alpha < 0$ in (6) under which, according to Definition 1, the nonlinear problem (1) generates mean-square contractive solutions. This issue is described in [12], as follows.

Definition 4. *Consider a nonlinear stochastic differential Eq. (1) satisfying assumptions (i)–(iii) given in Theorem 1 and let* X_n *and* Y_n*,* $n \geq 0$*, be two numerical solutions of (1) computed by the* ϑ*-methods (2) or (3). Then, the applied method is said to generate mean-square contractive numerical solutions in a region* $\mathcal{R} \subseteq \mathbb{R}^+$ *if, for a fixed* $\vartheta \in [0,1]$*,*

$$\nu(\vartheta,\Delta t) < 0, \quad \forall \Delta t \in \mathcal{R}$$

for (2), being $\nu(\vartheta,\Delta t)$ *the parameter in (14), or*

$$\epsilon(\vartheta,\Delta t) < 0, \quad \forall \Delta t \in \mathcal{R}$$

for (3), where $\epsilon(\vartheta,\Delta t)$ *is the parameter in (19).*

Definition 5. *A stochastic* ϑ*-method (2) or (3) is said unconditionally mean-square contractive if, for a given* $\vartheta \in [0,1]$*,* $\mathcal{R} = \mathbb{R}^+$*.*

As regards the ϑ-Maruyama method (2), according to Definition 4, mean-square contractive numerical solutions are generated if

$$0 < \beta(\vartheta,\Delta t) < 1,$$

for any $\Delta t \in \mathcal{R}$, i.e.

$$\mathcal{R} = \begin{cases} \left(0, \dfrac{|\alpha|}{(1-\vartheta)^2 M}\right), & \vartheta < 1, \\ \mathbb{R}^+, & \vartheta = 1. \end{cases} \tag{24}$$

As a consequence, the ϑ-Maruyama method with $\vartheta = 1$, i.e., the implicit Euler-Maruyama method

$$X_{n+1} = X_n + \Delta t f(X_{n+1}) + g(X_n)\Delta W_n. \tag{25}$$

is unconditionally mean-square contractive. In other terms, the stochastic perturbation (25) of the deterministic implicit Euler method preserves its unconditional contractivity property [18].

In analogous way, as regards the ϑ-Milstein method (3), Definition 4 leads to

$$0 < \gamma(\vartheta, \Delta t) < 1,$$

for any $\Delta t \in \mathcal{R}$, i.e.

$$\mathcal{R} = \begin{cases} \left(0, \dfrac{4|\alpha|}{4(1-\vartheta)^2 M + 3\widetilde{M}}\right), & \vartheta < 1, \\ \left(0, \dfrac{4|\alpha|}{3\widetilde{M}}\right), & \vartheta = 1. \end{cases} \tag{26}$$

Finally, the computation of the regions $\mathcal{R}$ in (24) and (26) require the knowledge of the parameters L, μ, M and $\widetilde{M}$. To make the region $\mathcal{R}$ completely computable, the authors in [12] have adopted the estimation strategy described in [34].

3 Optimal Choice of the Parameter ϑ

We now aim to investigate the choice of the optimal parameter ϑ, for both ϑ-methods (2) and (3), that maximizes the region $\mathcal{R}$ in (24) and (26), as well as the linear stability interval provided by (11) and (12).

To this purpose, let us denote by λ and σ the intercepts of the linearizations of the functions f and g in (1), respectively. In the remainder, we assume λ and σ to be real. In addition, for the nonlinear SDE (1), we assume the conditions of Theorem 1 satisfied, i.e., we deal with exponential mean-square dissipative nonlinear SDEs, according to the Definition 1.

3.1 Analysis of Stochastic ϑ-Maruyama Methdos

We first consider stochastic ϑ-Maruyama methods (2). We note from (11) that, in order to satisfy (10), the following condition must be fulfilled

$$\Delta t(1-2\vartheta)\lambda^2 < -2\left(\lambda + \frac{1}{2}\sigma^2\right). \tag{27}$$

Moreover, the stability condition for the linear SDE (7) provides that

$$\lambda + \frac{1}{2}\sigma^2 < 0.$$

It is worth separating the analysis of the two cases $\vartheta \in [0, \frac{1}{2})$ and $\vartheta \in [\frac{1}{2}, 1)$. Let us start with the case $\vartheta \in [0, \frac{1}{2})$. Taking into account (11) and (27), for $\vartheta \in [0, \frac{1}{2})$, we define

$$R_{EM} = \begin{bmatrix} -\dfrac{2\lambda + \sigma^2}{(1-2\vartheta)\lambda^2} \\ \\ \dfrac{|\alpha|}{(1-\vartheta)^2 M} \end{bmatrix}$$

and denote

$$h_{EM}(\vartheta) = ||R_{EM}||^2 = \frac{\left(2\lambda + \sigma^2\right)^2}{(1-2\vartheta)^2} + \frac{\alpha^2}{(1-\vartheta)^4 M^2},$$

whose first derivative is then given by

$$h'_{EM}(\vartheta) = \frac{\left(2\lambda + \sigma^2\right)^2}{(1-2\vartheta)^3} + \frac{4\alpha^2}{M^2\left(1-\vartheta\right)^5}. \tag{28}$$

Hence

$$h_{EM}(\vartheta) > 0, \qquad 0 \leq \vartheta < \frac{1}{2}. \tag{29}$$

Therefore, the function $||R_{EM}||$ is a increasing function of ϑ.

We now analyze the case $\frac{1}{2} \leq \vartheta < 1$. In this case, the linear stability condition (27) is satisfied for any $\Delta t > 0$. This means that the linear stability region is unbounded and the study of the term $||R_{EM}||$ reduces only in the analysis of

$$h_{EM}(\vartheta) = \frac{|\alpha|}{M(1-\vartheta)^2}, \qquad \frac{1}{2} \leq \vartheta < 1,$$

that is an increasing function of ϑ as well.

Finally, for $\vartheta = 1$, both the linear and the nonlinear stability regions are equal to $\mathbb{R}^+$. Hence, we argue that $\vartheta = 1$ is the optimal choice for ϑ in ϑ-Maruyama methods.

3.2 Analysis of Stochastic ϑ-Milstein Methods

Let us now analyze the family of stochastic ϑ-Milstein methods (3). Taking into account (12) and (13), proper algebraic manipulations suggest that, in order to attain (10), the following condition must hold true

$$\gamma(\vartheta)\Delta t < \left|2\lambda + \sigma^2\right|, \tag{30}$$

where

$$\gamma(\vartheta) = \frac{1}{2}\sigma^4 + \lambda\left(\lambda + 2\vartheta\left(\sigma^2 - \lambda\right) - 2\sigma^2\right). \tag{31}$$

If we define

$$l(\lambda,\sigma) = \frac{\frac{\sigma^4}{2|\lambda|} - \lambda + 2\sigma^2}{2\left(\sigma^2 - \lambda\right)},$$

we have

$$\gamma(\vartheta) > 0$$

if and only if

$$\vartheta < l(\lambda,\sigma).$$

Let us distinguish the following two cases.

Case $l(\lambda,\sigma) < 1$. In this case, for $0 \le \vartheta < l(\lambda,\sigma)$, we define

$$R_{MIL} = \begin{bmatrix} \dfrac{2\left|\lambda + \frac{1}{2}\sigma^2\right|}{\gamma(\vartheta)} \\ \\ \dfrac{4|\alpha|}{4(1-\vartheta)^2 M + 3\widetilde{M}} \end{bmatrix}$$

and

$$h_{MIL}(\vartheta) = ||R_{MIL}||^2 = \frac{4|\lambda + \frac{1}{2}\sigma^2|^2}{\gamma^2(\vartheta)} + \frac{16|\alpha|^2}{(4(1-\vartheta)^2 M + 3\widetilde{M})^2},$$

whose first derivative is given by

$$h'_{MIL}(\vartheta)\frac{-8|\lambda + \frac{1}{2}\sigma^2|^2\gamma'(\vartheta)}{\gamma^3(\vartheta)} + \frac{32\cdot 8|\alpha|^2(1-\vartheta)}{(4(1-\vartheta)^2 M + 3\widetilde{M})^3}. \tag{32}$$

Since, by (30), $\gamma'(\vartheta) < 0$ for any $0 \le \vartheta < l(\lambda,\sigma)$, we have that R_{MIL} is an increasing function of ϑ, $0 \le \vartheta < l(\sigma,\gamma)$.

Case $l(\lambda,\sigma) \ge 1$. In this case, the linear stability interval is unbounded. Hence, by the second summand in (32), we conclude that R_{MIL} is an increasing function of ϑ, $0 \le \vartheta \le 1$, as well. This shows that also for the stochastic ϑ-Milstein methods, the choice $\vartheta = 1$ is the optimal one.

3.3 A Numerical Evidence: Stochastic Ginzburg-Landau Equation

Ginzburg-Landau equation plays a relevant role in the theory of superconductivity, in particular to describe phase transitions (see [17,23] and references therein). Its stochastic counterpart in presence of multiplicative noise has been provided in [25]. Specifically, we consider the scalar SDE (1) with

$$f(X(t)) = kX(t) - \tau X(t)^3, \qquad g(X(t)) = \rho X(t), \tag{33}$$

being $k, \tau, \rho \in \mathbb{R}$. In our experiments, let us consider $k = -4$, $\tau = \rho = 1$, as in [12,22]. For this problem the constants L and μ are given by $L = 1$ and $\mu = -4$, so $\alpha = -7$. Then, according to Theorem 1, this problem generates mean-square contractive solutions. Moreover, the values of M in (17) and $\widetilde{M}$ in (22) are 16 and 1, respectively. We consider the stochastic ϑ-Maruyama method (2) with $\vartheta = 0.2$. In this case (24) yields

$$\mathcal{R} = (0,\ 0.68)\,.$$

The corresponding estimate on Δt is confirmed in Fig. 1, where the pattern of the mean-square deviation $\mathbb{E}|X_n - Y_n|^2$ in logarithmic scale is depicted for various values of Δt. It is visible that, the more Δt decreases, the more the numerical slope $\nu(0.2, \Delta t)$ in (14) tends to the exact one α in (6). For values of $\Delta t > 0.68$, the mean-square deviation does not exponentially decay.

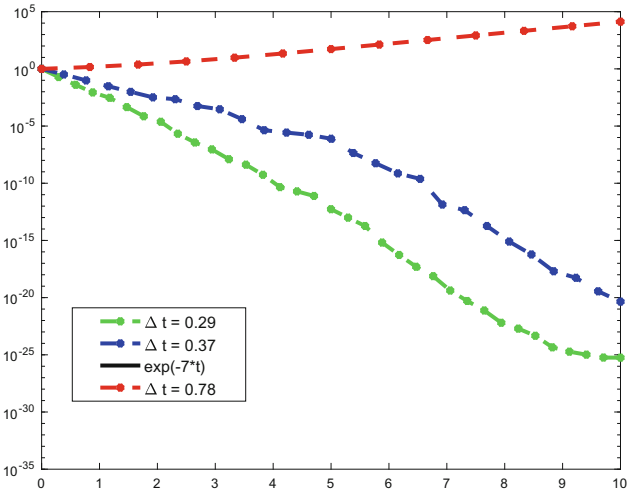

Fig. 1. Mean-square deviations over 2000 paths for the stochastic ϑ-method (2), with $\vartheta = 0.2$, applied to stochastic Ginzburg-Landau equation.

Let us now fix the value of stepsize $\Delta t = 400/2^9$ and perform the simulations over $M = 2000$ paths obtained applying the stochastic ϑ-Maruyama methods (2) and the stochastic ϑ-Milstein methods with $\vartheta = 0, 0.2, 0.5, 0.8, 1$. The numerical evidence provided in Figs. 2 and 3 reveals that, for decreasing values of the parameter ϑ, the mean-square contractivity of the relative stochastic ϑ-Maruyama method is less and less visible, according to the theory here provided.

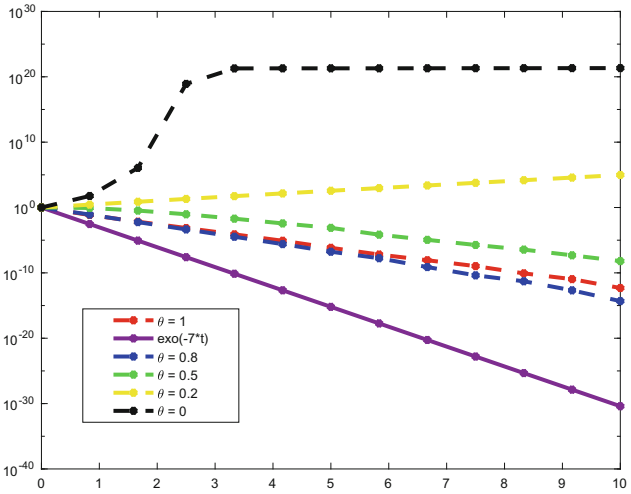

Fig. 2. Mean-square deviations over 2000 paths for stochastic ϑ-Maruyama methods (2), with different values of ϑ and constant stepsize $\Delta t = 400/2^9$, applied to the stochastic Ginzburg-Landau equation.

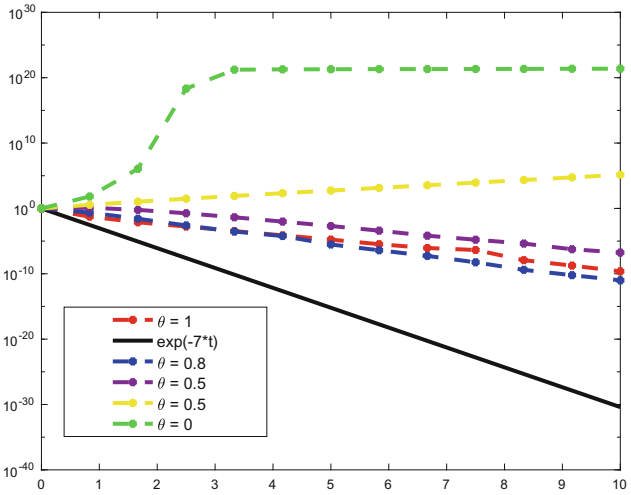

Fig. 3. Mean-square deviations over 2000 paths for stochastic ϑ-Mistein methods (3), with different values of ϑ and constant stepsize $\Delta t = 400/2^9$, applied to stochastic Ginzburg-Landau equation.

4 Conclusions and Future Issues

In this work, we have considered nonlinear stochastic differential equations (1) exhibiting an exponential mean-square dissipative character according to Theorem 1 and Definition 1. We have analyzed the behaviours of stochastic ϑ-methods, i.e., the ϑ-Maruyama (2) and ϑ-Milstein methods (3). As proved in [12], these discretizations are capable of reproducing the mean-square contractivity, according to Definition 4, in accordance to a suitable stepsize restriction. In this paper, we have discussed about to choice of the parameter ϑ in order to maximize both the linear and nonlinear stability intervals. It has been proved that the optimal choice is $\vartheta = 1$ for both families of ϑ-methods, as confirmed by Figs. 2 and 3.

Future developments of this research will be oriented to provide optimal ϑ-methods for SDEs exhibiting a mean-square dissipative character in presence of other types of noise and also in presence of jumps [21,22]. Moreover, the effectiveness of ϑ-methods and their extensions in numerically retaining the characteristic features of nonlinear stochastic oscillators [6,7,14,16,33] deserves further future investigations.

We also highlight that the analysis of nonlinear stability properties of other numerical methods (such as two-step Runge-Kutta methods [15]) also deserves future investigations.

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