Continuous Extension of Euler-Maruyama Method for Stochastic Differential Equations

Dajana Conte[1], Raffaele D'Ambrosio[2], Giuseppe Giordano[1(✉)], and Beatrice Paternoster[1]

[1] Department of Mathematics, University of Salerno, Fisciano, Italy
{dajconte,gigiordano,beapat}@unisa.it
[2] DISIM, University of L'Aquila, L'Aquila, Italy
raffaele.dambrosio@univaq.it

Abstract. In this work we focus on the development of continuous extension of Euler-Maruyama method, which is used to numerically approximate the solution of Stochastic Differential Equations (SDEs). We aim to provide an approximation of a given SDE in terms of a piecewise polynomial, because, as it is known in the deterministic case, a dense output allows to provide a more efficient error estimate and it is very effective for a variable step-size implementation. Hence, this contribution aims to provide a first building block in such directions, consisting in the development of the scheme.

Keywords: Stochastic Differential Equations · Euler-Maruyama method · Collocation methods · Dense output

1 Introduction

The study of Stochastic Differential Equations (SDEs) has attracted the recent attention of many researchers, due to their applications in many areas such as physics, finance, biology, medicine and chemistry [13,33,34,41,43,47]. The general form of an autonomous SDEs is given by:

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0, \quad t \in [0,T], \tag{1}$$

where the solution $X(t)$ is a random variable, the coefficient $f(X(t))$ of the deterministic part is the *drift coefficient*, while the coefficient $g(X(t))$ of the stochastic term is denoted as *diffusion coefficient* and $W(t)$ is a multidimensional Wiener process. We remind that a Wiener process $W(t)$ on $[0,T]$ is a real valued continuous-time process with the property that the Wiener increments $W(t) - W(s)$ are independent random variables with

$$W(t) - W(s) \sim \sqrt{t-s}N(0,1), \qquad 0 \le s < t \le T,$$

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where $N(0,1)$ denotes a normally distributed random variable with zero mean and unitary variance.

The integral form of (1) is given by

$$X(t) = X(0) + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s), \quad t \in [0,T], \tag{2}$$

where the first integral is a Riemann integral, while the second one is a stochastic integral, commonly interpreted in either Itô or Stratonovich form. In this paper, we will interpret the second integral of the Eq. (2) as an Itô integral. For a comprehensive treatise of stochastic calculus (also in the direction of numerical integration), as well as on the existence and the uniqueness of solutions to (2), the interested reader can refer, for instance, to [33,41,43,47] and references therein.

Numerical modeling for SDEs is a crucial ingredient to accurately simulate the solutions to nonlinear SDEs, since they cannot be exactly computed in most of the cases. Comprehensive monographs on the topic are, for instance, [33,41]. Relevant families of stochastic one-step, two-step, Runge-Kutta and two-step Runge-Kutta methods are discussed, for instance, in [20,32,33,39–41,43,47–49] and references therein. Moreover, contributions on the preservation of invariance laws along time discretizations are object of [12,13,13,19,26,28,29,31,38,42] and references therein.

With respect to the uniform grid

$$I_h = \{t_n = nh,\ n = 0,1,\ldots,n,\ Nh = T\}, \quad h > 0, \tag{3}$$

the well-known Euler-Maruyama method (EM)

$$X_j = X_{j-1} + hf(X_{j-1}) + g(X_{j-1})\Delta W_j, \quad j = 1,2,\ldots,N, \tag{4}$$

represents a prototype of the simplest method for SDEs (1). X_j is an approximate solution to $X(t_j)$ and $\Delta W_j = W(t_j) - W(t_{j-1})$ are independent Wiener increments. The present contribution aims to develop a continuous extension to the EM method, useful to provide a building block for a more general theory of continuous methods for SDEs, useful to assess their variable stepsize implementation, particularly effective, for instance, to solve stiff SDEs [1–7,9–11,14,21,35,36,44–46,50].

The idea we aim to use here is closely related to the analogous scenario of continuous numerical methods for deterministic differential equations at the basis, for instance, of the well-known idea of *deterministic numerical collocation* (refer, for instance, to [16–18,22–25,27,37] and references therein).

Numerical collocation for deterministic problems is a feasible technique to develop dense output methods for functional equations. In other terms, the provided approximant (the *collocation function*) is constructed as linear combination of selected basis functions, spanning a finite dimensional functional space; usually these functions are algebraic polynomials. The collocation function is required to exactly satisfies the given equation in a selected set points of the integration interval, denoted as *collocation points*. It is worth observing that

the collocation function can be chosen as a linear combination of ad hoc basis functions, chosen coherently with the qualitative character of the given problem.

Furthermore, the collocation function provides an approximation to the solution of the continuous problem over the entire integration interval. This feature is particularly, for instance, in order to provide high continuous order methods not suffering from order reduction when applied to stiff problems [15,25], as well as to provide efficient and accurate procedure of error estimation useful in variable stepsize implementations [30]. Nonetheless, re-casting discrete numerical methods as collocation methods makes their analysis benefit from their continuous formulation.

Inspired by the idea of collocation for Volterra integral equations [8], we aim to provide here a continuous extension of the EM method (4) applied to the integral form (2) of the SDE (1). The paper is organized as follows: in Sect. 2 we briefly recall the idea of numerical collocation for Volterra Integral Equations; this technique is extended to stochastic differential equations and, for selected values of the collocation parameters, the extension of the EM method (4) is provided. Some concluding remarks are discussed in Sect. 5.

2 Numerical Collocation for Volterra Integral Equations

This section aims to recall the idea of collocation for deterministic Volterra Integral Equations (VIEs)

$$y(t) = g(t) + \int_0^t k(t, \tau, y(\tau))d\tau, \quad t \in I = [0, T], \tag{5}$$

where $k \in C(D \times R)$, with $D = \{(t, \tau) : 0 \leq \tau \leq t \leq T\}$ and $g \in C(t)$.
Let us refer to the set of grid points I_h defined in (3) and let

$$0 \leq c_1 < \ldots < c_m \leq 1$$

be m collocation parameters. Moreover, let us denote by $t_{nj} = t_n + c_j h$ the so-called collocation points. Collocation methods allow us to approximate the exact solution $y(t)$ of (5) by a piecewise polynomial

$$P(t) \in S_{m-1}^{(-1)}(I_h) = \{v|_{(t_n, t_{n+1}} \in \Pi_{m-1}, n = 0, 1, \ldots, N-1\},$$

where Π_{m-1} denotes the space of algebraic polynomials of degree not exceeding $m-1$. The collocation polynomial, restricted to the interval $[t_n, t_{n+1}]$, is of the form

$$P_n(t_n + sh) = \sum_{j=1}^{m} L_j(s) Y_{nj}, \quad s \in [0, 1], \tag{6}$$

for $n = 0, \ldots, N-1$, where $L_j(s)$ is the j-th Lagrange fundamental polynomial and $Y_{nj} = P_n(t_{nj})$. The collocation equation is obtained by imposing that the polynomial (6) exactly satisfies the integral equation in the collocation points

and requires, at each time step, the solution of a system of m nonlinear equations in the m unknowns Y_{ni}, as follows

$$\begin{cases} Y_{ni} = F_{ni} + \Phi_{ni}, \\ y_{n+1} = \sum_{j=1}^{m} L_j(1) Y_{nj}, \end{cases},$$

where

$$F_{ni} = g(t_{ni}) + h \sum_{v=0}^{n-1} \int_0^1 k\left(t_{ni}, t_v + sh, P_v(t_v + sh)\right) ds, \tag{7}$$

$$\Phi_{ni} = h \int_0^{c_i} k\left(t_{ni}, t_n + sh, P_n(t_n + sh)\right) ds, \tag{8}$$

for $i = 1, \ldots, m$.

In order to obtain a fully discretized collocation scheme, it is necessary to approximate the integrals appearing in (7) and (8) by suitable quadrature formulae. This procedure leads to the well-known idea of discretized collocation. The discretized collocation polynomial is of the form

$$\tilde{P}_n(t_n + sh) = \sum_{j=1}^{m} L_j(s) \tilde{Y}_{nj}, \quad s \in [0, 1], \tag{9}$$

for $n = 0, \ldots, N-1$, where $\tilde{Y}_{nj} = \tilde{P}_n(t_{nj})$. The corresponding discretized collocation method assume the form:

$$\begin{cases} \tilde{Y}_{ni} = \tilde{F}_{ni} + \tilde{\Phi}_{ni}, \\ \tilde{y}_{n+1} = \sum_{j=1}^{m} L_j(1) \tilde{Y}_{nj}, \end{cases},$$

where

$$\tilde{F}_{ni} = g(t_{ni}) + h \sum_{v=0}^{n-1} \sum_{l=0}^{\mu_1} b_l k\left(t_{ni}, t_v + \xi_l h, \tilde{P}_v(t_v + \xi_l h)\right),$$

$$\tilde{\Phi}_{ni} = h \sum_{l=0}^{\mu_2} w_{il} k\left(t_{ni}, t_n + d_{il} h, \tilde{P}_n(t_n + d_{il} h)\right),$$

for $i = 1, \ldots, m$, where

$$(\xi_l, b_l)_{l=1}^{\mu_1} \qquad (d_{il}, w_{il})_{l=1}^{\mu_2}$$

are two quadrature formulas with nodes ξ_l and d_{il} such that

$$0 \le \xi_1 < \ldots < \xi_{\mu_1} \le 1$$

and

$$0 \le d_{i1} < \ldots < d_{i\mu_2} \le 1.$$

μ_1 and μ_2 are positive integers and b_l and w_{il} are the weights of the formulae.

3 Continuous Extension of the EM Method

Let us now focus our attention to the SDE in integral form (2). We aim to provide an approximation $u(t)$ to the solution $X(t)$ of (2) on the overall integration interval $[0, T]$, such that its restriction $u_n(t)$ to the interval $[t_n, t_{n+1}]$ is a linear polynomial.

To this purpose, let us fix the collocation parameters $c_1, c_2 \in [0, 1]$ and denote by

$$t_{n1} = t_n + c_1 h,$$
$$t_{n2} = t_n + c_2 h$$

the two corresponding collocation points in the interval $[t_n, t_{n+1}]$.

Let us make the following ansatz: in a sufficiently small interval of length h, the solution $X(t)$ to (2) can be approximated by the linear function

$$u_n(t_n + \theta h) = L_1(\theta) U_1^{[n]} + L_2(\theta) U_2^{[n]}, \quad \theta \in [0, 1], \tag{10}$$

where $L_j(\theta)$ are the Lagrange fundamental polynomials with respect the collocation parameters, i.e.,

$$L_1(\theta) = \frac{\theta - c_2}{c_1 - c_2},$$
$$L_2(\theta) = \frac{\theta - c_1}{c_2 - c_1}$$

and $U_i^{[n]} = u_n(t_{ni})$, $i = 1, 2$.

Correspondingly, the following numerical scheme is obtained

$$\begin{cases} U_i^{[n]} = F_n + \Phi_i^{[n]}, \quad i = 1, 2, \\ X_{n+1} = L_1(1) U_1^{[n]} + L_2(1) U_2^{[n]}, \end{cases} \tag{11}$$

where the lag-term

$$F_n = X(t_0) + \int_0^{t_n} f(u(s)) ds + \int_0^{t_n} g(u(s)) dW(s), \tag{12}$$

contains all the past history of the dynamics up to the grid point t_n, while the incremental terms $\Phi_1^{[n]}$ and $\Phi_2^{[n]}$ are defined by

$$\Phi_i^{[n]} = \int_{t_n}^{t_{ni}} f(u_n(s)) ds + \int_{t_n}^{t_{ni}} g(u_n(s)) dW(s).$$

By employing (10) and by a suitable change of variable, the incremental terms assume the form

$$\Phi_1^{[n]} = h \int_0^{c_1} f\left(\sum_{j=1}^{2} L_j(\theta) U_j^{[n]} \right) d\theta + \int_{t_n}^{t_{n1}} g\left(\sum_{j=1}^{2} L_j\left(\frac{s - t_n}{h} \right) U_j^{[n]} \right) dW(s), \tag{13}$$

$$\Phi_2^{[n]} = h \int_0^{c_2} f\left(\sum_{j=1}^{2} L_j(\theta) U_j^{[n]}\right) d\theta + \int_{t_n}^{t_{n2}} g\left(\sum_{j=1}^{2} L_j\left(\frac{s-t_n}{h}\right) U_j^{[n]}\right) dW(s). \tag{14}$$

We observe that the scheme (11) requires the solution of a system of two nonlinear equations in the unknowns $U_1^{[n]}$ and $U_2^{[n]}$.

By setting $c_2 = 1$, the scheme (11) may be substantively simplified. In fact, in this case, we observe that

$$\Phi_2^{[n]} = \int_{t_n}^{t_{n+1}} f(u_n(s))ds + \int_{t_n}^{t_{n+1}} g(u_n(s))dW(s), \tag{15}$$

therefore, by (11), (12) and (15),

$$U_n^{[2]} = F_n + \Phi_2^{[n]} = F_{n+1}, \quad n = 0, \ldots, N-1.$$

As a consequence, by (10),

$$F_n = U_2^{[n-1]} = u_{n-1}(t_{n-1} + c_2 h) = u_{n-1}(t_n) = X_n,$$

i.e. for each time step the lag term F_n is equal to the numerical solution X_n at the mesh point t_n. Moreover,

$$X_{n+1} = u_n(t_n + h) = u_n(t_n + c_2 h) = U_2^{[n]}$$

and the method (11) assumes the form

$$\begin{cases} U_i^{[n]} = X_n + \Phi_i^{[n]}, & i = 1, 2, \\ X_{n+1} = U_2^{[n]}, \end{cases} \tag{16}$$

where the expressions of $\Phi_i^{[n]}$ are the same of (13) and (14), for $i = 1, 2$ respectively.

Clearly, the scheme reported in (11) is not a full discretization, unless the involved integrals are replaced by suitable quadrature formulae.

Remaining in the case $c_2 = 1$, we consider the following discretized metod, which is obtained from (16) by approximating the integrals in (13) and (14) by means of the rectangular quadrature rule

$$\begin{cases} \tilde{U}_1^{[n]} = \tilde{X}_n + h c_1 f\left(b_1 \tilde{U}_1^{[n]} + b_2 \tilde{U}_2^{[n]}\right) + g\left(b_1 \tilde{U}_1^{[n]} + b_2 \tilde{U}_2^{[n]}\right) \sqrt{c_1 h} Z_1^{[n]}, \\ \tilde{U}_2^{[n]} = \tilde{X}_n + h f\left(b_1 \tilde{U}_1^{[n]} + b_2 \tilde{U}_2^{[n]}\right) + g\left(b_1 \tilde{U}_1^{[n]} + b_2 \tilde{U}_2^{[n]}\right) \sqrt{h} Z_2^{[n]}, \\ \tilde{X}_{n+1} = \tilde{U}_2^{[n]}, \end{cases} \tag{17}$$

where $b_1 = L_1(0)$, $b_2 = L_2(0)$ and $Z_1^{[n]}$, $Z_2^{[n]}$ are normal variables of mean 0 and variance 1. The continuous approximant, in the interval $[t_n, t_{n+1}]$, assumes the form

$$\tilde{u}_n(t_n + \theta h) = \frac{1-\theta}{1-c_1} \tilde{U}_1^{[n]} + \frac{\theta - c_1}{1-c_1} \tilde{U}_2^{[n]} \quad \theta \in [0, 1]. \tag{18}$$

The derived family of continuous methods contains the EM method as a particular case when $c_1 = 0$. As a matter of fact, imposing $c_1 = 0$ in (17) and taking into account that $b_1 = 1$ and $b_2 = 0$, the method assumes the form

$$\begin{cases} \tilde{U}_1^{[n]} = \tilde{X}_n, \\ \tilde{U}_2^{[n]} = \tilde{X}_n + hf(\tilde{X}_n) + g(\tilde{X}_n)\sqrt{h}Z_2^{[n]} \end{cases}$$

and the value of $\tilde{X}_{n+1}$ can be evaluated as

$$\tilde{X}_{n+1} = \tilde{X}_n + hf(\tilde{X}_n) + g(\tilde{X}_n)\sqrt{h}Z_2^{[n]}, \tag{19}$$

which corresponds to the EM method (4). In the case of EM method the continuous approximant assumes the form

$$\tilde{u}_n(t_n + \theta h) = (1-\theta)\tilde{X}_n + \theta\tilde{X}_{n+1} \quad \theta \in [0,1]. \tag{20}$$

4 Numerical Evidences

In this section we experimentally observe the accuracy of the new class of methods obtained in the previous section, with special emphasis its weak accuracy. We recall that a numerical method has weak order of convergence equal to p if there exists a constant C such that

$$|\mathbb{E}(X_n) - \mathbb{E}(X(t_n))| \leq Ch^p$$

for any fixed $t_n = nh \in I_h$, where $\mathbb{E}(\cdot)$ is the expectation operator.

Let us assume the following SDE as test problem

$$dX(t) = t^2 dt + e^{\frac{d}{2}}\cos(X(t)), \quad X(0) = 0, \quad t \in [0,1]. \tag{21}$$

The expected value of the exact solution is $\mathbb{E}(X(t)) = \dfrac{t^3}{3}$.

Table 1. Absolute errors in the endpoint $T = 1$ obtained applying (11) to problem (21), for several values of the parameter c_1 .

N	$c_1 = 0$	$c_1 = 1/4$	$c_1 = 1/2$	$c_1 = 3/4$
2^7	$3.0967 \cdot 10^{-3}$	$3.0675 \cdot 10^{-3}$	$3.0095 \cdot 10^{-3}$	$2.8355 \cdot 10^{-3}$
2^6	$6.5909 \cdot 10^{-3}$	$6.5634 \cdot 10^{-3}$	$6.5083 \cdot 10^{-3}$	$6.3435 \cdot 10^{-3}$
2^5	$1.5071 \cdot 10^{-2}$	$1.5044 \cdot 10^{-2}$	$1.5071 \cdot 10^{-2}$	$1.4834 \cdot 10^{-2}$
2^4	$2.8721 \cdot 10^{-2}$	$2.8698 \cdot 10^{-2}$	$2.8653 \cdot 10^{-2}$	$2.8520 \cdot 10^{-2}$

Table 1 shows the absolute errors corresponding to various numerical methods belonging to the family (11), for selected values of c_1 and setting $c_2 = 1$, applied

to problem (21). All methods exhibit the same weak order of convergence of EM method (4), which is equal to 1. In other terms, passing to the continuous extension do not deteriorate the order EM.

Let us now set $c_1 = \frac{1}{4}$ and $c_2 = 1$, and apply the corresponding method to the following additional SDE

$$dX(t) = e^{-\frac{t}{2}}dt + e^{X(t)-\frac{t}{2}}dW(t), \quad X(0) = 0, \quad t \in [0,1], \tag{22}$$

whose exact solution has expected value $\mathbb{E}(X(t)) = 2-2e^{-\frac{t}{2}}$, and to the problem

$$d(X(t)) = (2X(t) + e^{2t})dt + \mu X(t)dW(t), \quad X(0) = 1, \quad t \in [0,1], \tag{23}$$

whose exact solution has expectation given by $\mathbb{E}(X(t)) = -e^{2t}(1+t)$.

Figure 1 shows the behaviour of the absolute errors for decreasing values of the stepsize. We can observe that the method obtained by imposing $c_1 = 1/4$, applied to (21), (22) and (23), exhibits weak order of convergence equal to 1. So, also for values of c_1 different from 0, we preserve the same weak order of convergence of EM method (4).

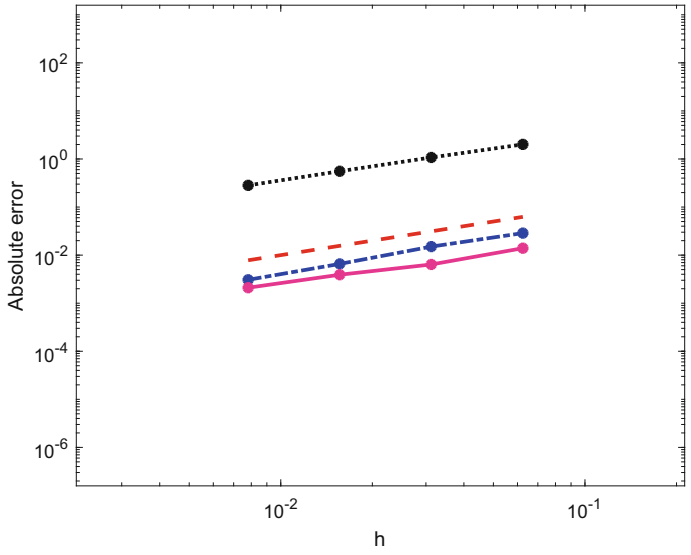

Fig. 1. Absolute weak errors associated to the application of (11) with $c_1 = 1/4$ to problem (21) (dashed-dotted blue line), problem (22) (solid magenta line) and problem (23) (dotted black line). The reference slope of order 1 is depicted by the dashed red line (Color figure online).

5 Concluding Remarks

We have focused on the possibility to continuously extend the EM method (4), by properly revising the idea of numerical collocation method used in the literature for deterministic Volterra integral equations. This continuous extension provides an approximation to the solution in the overall interval of integration.

The present contribution is a first building block for a comprehensive theory of continuous numerical methods for SDEs, useful to assess a variable stepsize framework particularly effective, for instance, to solve stiff SDEs [1–7,9–11,21,35,36,44–46,50]. In this direction, further efforts are requested, such as the construction of feasible error estimates by exploiting the continuous approximant. Basis functions other than algebraic polynomials will also be investigated.

References

1. Abdulle, A.: Explicit methods for stiff stochastic differential equations. In: Engquist, B., Runborg, O., Tsai, Y,H. (eds.) Numerical Analysis of Multiscale Computations, LNCSE, vol. 82, pp. 1–22. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-21943-6_1
2. Abdulle, A., Almuslimani, I., Vilmart, G.: Optimal explicit stabilized integrator of weak order 1 for stiff and ergodic stochastic differential equations. SIAM-ASA J. Uncertain. **6**(2), 937–964 (2018)
3. Abdulle, A., Blumenthal, A.: Stabilized multilevel Monte Carlo method for stiff stochastic differential equations. J. Comput. Phys. **251**, 445–460 (2013)
4. Abdulle, A., Cirilli, S.: S-ROCK: Chebyshev methods for stiff stochastic differential equations. SIAM J. Sci. Comput. **30**(2), 997–1014 (2007)
5. Abdulle, A., Cirilli, S.: Stabilized methods for stiff stochastic systems. C. R. Math. **345**(10), 593–598 (2007)
6. Abdulle, A., Vilmart, G., Zygalakis, K.C.: Mean-square A-stable diagonally drift-implicit integrators of weak second order for stiff Itô stochastic differential equations. BIT Numer. Math. **53**(4), 827–840 (2013)
7. Abdulle, A., Vilmart, G., Zygalakis, K.C.: Weak second order explicit stabilized methods for stiff stochastic differential equations. SIAM J. Sci. Comput. **35**(4), A1792–A1814 (2013)
8. Brunner, H.: Collocation Methods for Volterra Integral and Related Functional Equations. Cambridge University Press, Cambridge (2004)
9. Burrage, K., Tian, T.: Stiffly accurate Runge-Kutta methods for stiff stochastic differential equations. Comput. Phys. Commun. **142**(1–3), 186–190 (2001)
10. Burrage, K., Burrage, P.M.: A variable stepsize implementation for stochastic differential equations. SIAM J. Sci. Comput. **24**(3), 848–864 (2003)
11. Burrage, K., Burrage, P.M., Herdiana, R.: Adaptive stepsize based on control theory for stochastic differential equations. J. Comput. Appl. Math. **170**(2–2), 317–336 (2004)
12. Burrage, K., Burrage, P.M.: Low rank Runge-Kutta methods, symplecticity and stochastic Hamiltonian problems with additive noise. J. Comput. Appl. Math. **236**(16), 3920–3930 (2012)
13. Burrage, P.M., Burrage, K.: Structure-preserving Runge-Kutta methods for stochastic Hamiltonian equations with additive noise. Numer. Algorithms **65**(3), 519–532 (2013). https://doi.org/10.1007/s11075-013-9796-6

14. Burrage, K., Tian, T.: The composite Euler method for stiff stochastic differential equations. J. Comput. Appl. Math. **131**(1–2), 407–426 (2001)
15. Butcher, J.C.: Numerical Methods for Ordinary Differential Equations, 3rd edn. Wiley, Chichester (2016)
16. Capobianco, G., Conte, D., Paternoster, B.: Construction and implementation of two-step continuous methods for Volterra Integral Equations. Appl. Numer. Math. **119**, 239–247 (2017)
17. Cardone, A., Conte, D.: Multistep collocation methods for Volterra integro-differential equations. Appl. Math. Comput. **221**, 770–785 (2013)
18. Cardone, A., Conte, D., D'Ambrosio, R., Paternoster, B.: Collocation methods for Volterra integral and integro-differential equations: a review. Axioms **7**(3), 45 (2018)
19. Chen, C., Cohen, D., D'Ambrosio, R., Lang, A.: Drift-preserving numerical integrators for stochastic Hamiltonian systems. Adv. Comput. Math. **46**(2), 1–22 (2020). https://doi.org/10.1007/s10444-020-09771-5
20. Citro, V., D'Ambrosio, R., Di Giovacchino, S.: A-stability preserving perturbation of Runge-Kutta methods for stochastic differential equations. Appl. Math. Lett. **102**, 106098 (2020)
21. Cohen, D., Sigg, M.: Convergence analysis of trigonometric methods for stiff second-order stochastic differential equations. Numer. Math. **121**(1), 1–29 (2012)
22. Conte, D., Del Prete, I.: Fast collocation methods for Volterra integral equations of convolution type. J. Comput. Appl. Math. **196**(2), 652–663 (2006)
23. Conte, D., Paternoster, B.: Multistep collocation methods for Volterra integral equations. Appl. Numer. Math. **59**(8), 1721–1736 (2009)
24. Conte, D., D'Ambrosio, R., Paternoster, B.: Two-step diagonally-implicit collocation based methods for Volterra Integral Equations. Appl. Numer. Math. **62**(10), 1312–1324 (2012)
25. D'Ambrosio, R., Ferro, M., Jackiewicz, Z., Paternoster, B.: Two-step almost collocations methods for ordinary differential equations. Numer. Algorithms **53**(2–3), 195–217 (2010)
26. D'Ambrosio, R., Giordano, G., Paternoster, B., Ventola, A.: Perturbative analysis of stochastic Hamiltonian problems under time discretizations. Appl. Math. Lett. **120**, 107223 (2021)
27. D'Ambrosio, R., Paternoster, B.: Multivalue collocation methods free from order reduction. J. Comput. Appl. Math. **387**, 112515 (2021)
28. D'Ambrosio, R., Di Giovacchino, S.: Mean-square contractivity of stochastic ϑ-methods. Commun. Nonlinear Sci. Numer. Simul. **96**, 105671 (2021)
29. D'Ambrosio, R., Di Giovacchino, S.: Nonlinear stability issues for stochastic Runge-Kutta methods. Commun. Nonlinear Sci. Numer. Simul. **94**, 105549 (2021)
30. D'Ambrosio, R., Jackiewicz, Z.: Construction and implementation of highly stable two-step continuous methods for stiff differential systems. Math. Comput. Simul. **81**(9), 1707–1728 (2011)
31. D'Ambrosio, R., Moccaldi, M., Paternoster, B.: Numerical preservation of long-term dynamics by stochastic two-step methods. Discr. Cont. Dyn. Sys. Ser. B **23**(7), 2763–2773 (2018)
32. D'Ambrosio, R., Scalone, C.: Two-step Runge-Kutta methods for stochastic differential equations. Appl. Math. Comput. **403**, 125930 (2021)
33. Gard, T.C.: Introduction to Stochastic Differential Equations. Marcel Dekker Inc., New York-Basel (1988)
34. Gardiner, C.Q.: Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences. Springer Series in Synergetics. Springer, Heidelberg (2009)

35. Haghighi, A., Hosseini, S.M., Rössler, A.: Diagonally drift-implicit Runge-Kutta methods of strong order one for stiff stochastic differential systems. J. Comput. Appl. Math. **293**, 82–93 (2016)
36. Haghighi, A., Rössler, A.: Split-step double balanced approximation methods for stiff stochastic differential equations. Int. J. Comp. Math. **96**(5), 1030–1047 (2019)
37. Hairer, E., Wanner, G.: Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems. Springer, Heidelberg (1996). https://doi.org/10.1007/978-3-642-05221-7
38. Han, M., Ma, Q., Ding, X.: High-order stochastic symplectic partitioned Runge-Kutta methods for stochastic Hamiltonian systems with additive noise. Appl. Math. Comput. **346**, 575–593 (2019)
39. Higham, D.J.: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. **43**(3), 525–546 (2001)
40. Higham, D.J.: Mean-square and asymptotic stability of the stochastic theta method. SIAM J. Numer. Anal. **38**(3), 753–769 (2000)
41. Higham, D., Kloeden, P.E.: An Introduction to the Numerical Simulation of Stochastic Differential Equations. SIAM, Philadelphia (2021)
42. Hong, J., Xu, D., Wang, P.: Preservation of quadratic invariants of stochastic differential equations via Runge-Kutta methods. Appl. Numer. Math. **87**, 38–52 (2015)
43. Kloeden, P.E., Platen, E.: The Numerical Solution of Stochastic Differential Equations. Springer, Heidelberg (1992). https://doi.org/10.1007/978-3-662-12616-5
44. Komori, Y., Burrage, K.: A stochastic exponential Euler scheme for simulation of stiff biochemical reaction systems. BIT Numer. Math. **54**(4), 1067–1085 (2014). https://doi.org/10.1007/s10543-014-0485-1
45. Li, T., Abdulle, A., Weinan, E.: Effectiveness of implicit methods for stiff stochastic differential equations. Commun. Comp. Phys. **3**(2), 295–307 (2008)
46. Milstein, G.N., Platen, E., Schurz, H.: Balanced implicit methods for stiff stochastic systems. SIAM J. Numer. Anal. **35**(3), 1010–1019 (1998)
47. Milstein, G.N., Tretyakov, M.V.: Stochastic Numerics for Mathematical Physics. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-662-10063-9
48. Rössler, A.: Runge-Kutta methods for Itô stochastic differential equations with scalar noise. BIT Numer. Math. **46**(1), 97–110 (2006)
49. Tang, X., Xiao, A.: Efficient weak second-order stochastic Runge-Kutta methods for Itô stochastic differential equations. BIT Numer. Math. **57**, 241–260 (2017)
50. Tian, T., Burrage, K.: Implicit Taylor methods for stiff stochastic differential equations. Appl. Numer. Math. **38**(1–2), 167–185 (2001)