New Package in *Maxima* to Build Axonometric Projections from $\mathbb{R}^4$ to $\mathbb{R}^3$ and Visualize Objects Immersed in $\mathbb{R}^4$

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Abstract. The human being has the need to represent the objects that surround him. But the world around us constitutes a three-dimensional reality and the formats in which it is represented are two-dimensional. Then the problem arises of representing on paper, which has two dimensions, any object immersed in a space that has three dimensions. In response to the problem, the Descriptive Geometry and the Representation Systems are born. The representation systems are a set of operations that allow make projections of objects immersed in three-dimensional space on a plane that is usually the role of drawing. A class of these systems are those obtained from axonometric projections of $\mathbb{R}^3$ to $\mathbb{R}^2$ based on Pohlke's theorem and widely used in the vast majority of scientific texts. In this paper it is proposed to build axonometric projections from $\mathbb{R}^4$ to $\mathbb{R}^3$ to obtain projections of objects immersed in the four-dimensional space on a 3D hyperplane. To visualize the results, a new package encoded in the *Maxima* open source software will be used.

Keywords: Pohlke theorem · Representation systems · Axonometric projections · Plot objetcs in $\mathbb{R}^4$

1 Introduction

The concept of multidimensionality becomes interesting, in the modern world, in the Victorian era [15]. It is at this time, exactly in 1884, that the English ecclesiastic Edwin Abbott [9] publishes the novel Flatland [1] in which themes of profound mathematical significance, related to the n-dimensional Euclidean-spaces (n from one to four) that are still difficult to assimilate in the current era, are described in a simple and enjoyable way. With the appearance of the first programmable electromechanical computers between the years 1930 and 1940 [3] and its subsequent development it becomes possible to venture into the visualization of objects of the tetradimensional Euclidean space and the first

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investigations are developed [4,8]. From these investigations many more have been carried out [12,16,18–21,23].

The foregoing paragraph indicates that the human being has the need to represent the objects that surround him or those he imagines or projects, to make them known to others. But the world around us constitutes a three-dimensional reality and the formats in which it is represented are two-dimensional. Then the problem arises of representing on paper, whose dimension is two, any object that is immersed in a three-dimensional space. In response to the problem, Descriptive Geometry and Representation Systems are born [10]. A class of these systems are those obtained from axonometric projections of $\mathbb{R}^3$ to $\mathbb{R}^2$ based on the geometric statement of the painter and professor of descriptive geometry, of German nationality, Karl Wilhelm Pohlke. Such a statement is fundamental to axonometric projections and is known as Pohlke's theorem. Pohlke formulated this theorem in 1853 and published it in 1860, without demonstration, in the first part of his textbook on descriptive geometry. The first elementary rigorous proof was given by H. A. Schwarz in 1864, at that time a student of Pohlke [7]. The most common of the axonometric projections is the one that considers three coincident segments at one point (all belonging to the same plane) such that two segments form an angle of ninety sexagesimal degrees and the third forms an angle of one hundred thirtyfive sexagesimal degrees with each of the first two segments [6].

In this paper it is proposed to build axonometric projections from $\mathbb{R}^4$ to $\mathbb{R}^3$ to obtain projections of objects immersed in the four-dimensional space on a 3D hyperplane. The limitation for this proposal to be understood as a natural extension of what is established in Pohlke's theorem is that there are no three-dimensional "screens", since all current computer screens are flat. However, the advantages offered by the *Maxima* v.5.43.0 will be used to simulate the visualization of objects immersed in $\mathbb{R}^3$ [17].

The structure of this paper is as follows: Sect. 2 introduce the Pohlke's theorem, some concepts of planar geometric projections, axonometry and the construction of axonometric projections from $\mathbb{R}^3$ to $\mathbb{R}^2$ and from $\mathbb{R}^4$ to $\mathbb{R}^3$. Then, Sect. 3 introduces the new *Maxima* package, `R4projections`, and describes the commands implemented within. Finally, Sect. 4 closes with the conclusions and recommendations.

2 Preliminaries

2.1 The Pohlke's Theorem

Theorem 1. Three lines of any length that originate from a point on a plane at different angles can always be considered as a parallel (biased) projection of three lines of the same length perpendicular to each other [13].

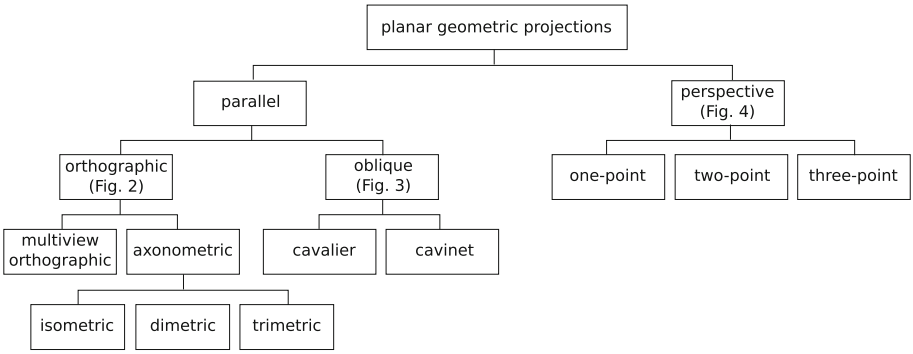

Fig. 1. Classification of projections [2].

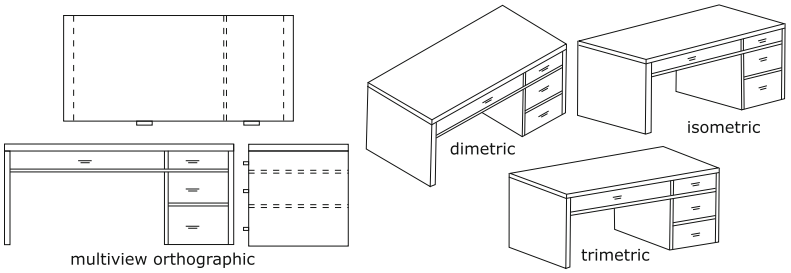

Fig. 2. Pictorial effects of orthographic projections [2].

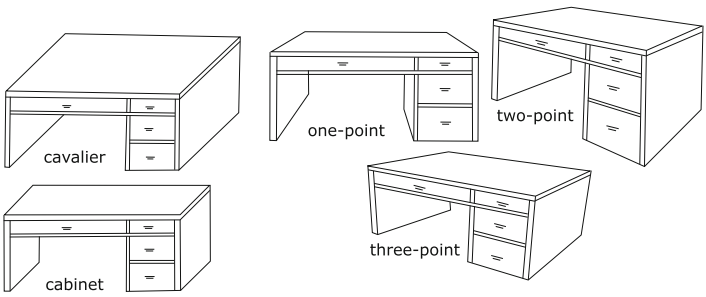

Fig. 3. Pictorial effects of oblique and perspective projections [2].

2.2 Concepts of Planar Geometric Projections

We will understand by *projection*[1] both the mapping of a three-dimensional space in a two-dimensional subspace and the image resulting from applying such a mapping to an object. Planar geometric projections can be classified

[1] This paper will work only with linear projections, specifically with axonometric projections.

into parallel projections or perspective projections; which in turn admit other classifications according to certain characteristics (Figs. 1, 2 and 3) [2].

2.3 Axonometric Projections from $\mathbb{R}^3$ to $\mathbb{R}^2$

Given a projection, by Pohlke's theorem, it is possible to select the projections of the coordinate axes as shown in Fig. 4, for example. In this way the $\bar{O}\bar{X}\bar{Y}\bar{Z}$ system will have been obtained as a projection of the $OXYZ$ system (Fig. 4). So, the image $\bar{P}$ of a point $P = (x, y, z)$ is determining by the four steps, starting a point $\bar{O}$: 1) go x in $\bar{X}$-direction, then 2) go y in $\bar{Y}$-direction, then 3) go z in $\bar{Z}$-direction and 4) mark the point as $\bar{P}$ (Fig. 4, right). It is worth mentioning that it is recommended to multiply the values x, y, z by factors

$$0 < v_x, v_y, v_z \leq 1 \tag{1}$$

in order to obtain a better representation [2].

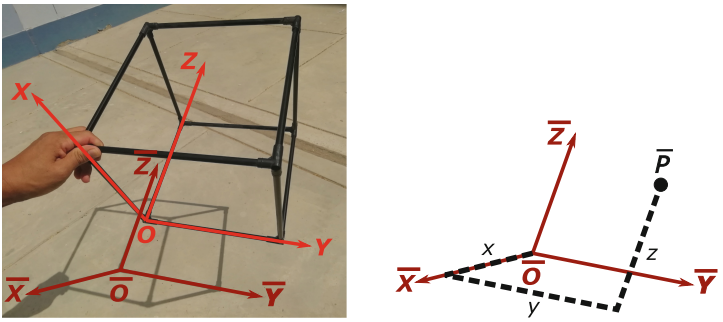

Fig. 4. Pohlke's theorem.

Figure 4 (right) shows a projection for which the arrangement of $\bar{X}, \bar{Y}, \bar{Z}$ axes is not the most common. Next, the analytical expressions of a projection that is the most common in most mathematical texts will be established (Fig. 5, left). To do this, the coordinate system generated by $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, with axes: OX, OY, OZ, will be projected on the coordinate system generated by $\{O'; \boldsymbol{e}_1, \boldsymbol{e}_2\}$, with axes: $O'X', O'Y'$ [11].

Let's define the projection $\mathbf{p} : \mathbb{R}^3 \to \mathbb{R}^2$ (Fig. 5) for which it is true that

$$\begin{aligned}
\mathbf{p}(\boldsymbol{i}) &= -\frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \\
\mathbf{p}(\boldsymbol{j}) &= \boldsymbol{e}_1 = (1, 0), \\
\mathbf{p}(\boldsymbol{k}) &= \boldsymbol{e}_2 = (0, 1).
\end{aligned}$$

Let $P = (x, y, z)$ be an arbitrary point of $\mathbb{R}^3$ then, considering a result of linear algebra,

$$\mathbf{p}(P) = \mathbf{p}(x, y, z) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2}$$

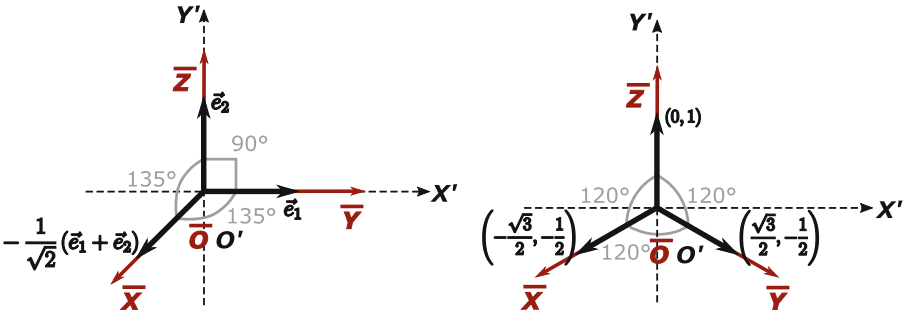

Fig. 5. Scheme to obtain the analytical expressions of two axonometric projection: dimetric (left) and isometric (right).

Equation (2) indicates that the point $P = (x, y, z)$ projects onto the point $P' = \left(-\frac{x}{\sqrt{2}} + y, -\frac{x}{\sqrt{2}} + z\right)$. This projection is called a dimetric projection.

Thus, for example, if another common axonometric projection is considered (Fig. 5, right), it is obtained

$$\mathbf{p}(x, y, z) = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{3}$$

This projection is called a isometric projection.

Next, the graphs of the projections of some mathematical objects will be shown. First, when applying a dimetric axometric projection (Fig. 6) and then when applying an isometric axonometric projection (Fig. 7). Note that, in the case of the axonometric projection, according to (1), $v_x = \frac{3}{5}$, $v_y = v_z = 1$; while for the other case, $v_x = v_y = v_z = 1$.

The analytical expressions of a projection $\mathbf{p}$ allow to obtain projections of mathematical objects from $\mathbb{R}^3$ to $\mathbb{R}^2$, according to the choice of $\bar{X}, \bar{Y}, \bar{Z}$ axes.

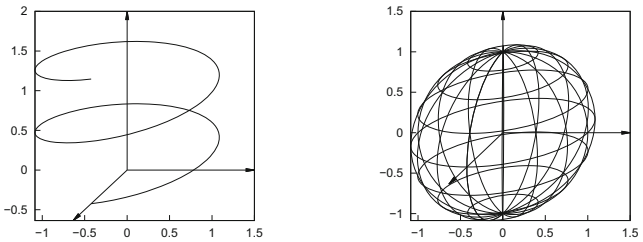

Fig. 6. Dimetric axonometric projection of a circular helix (left) as well as the u-parameter and v-parameter curves of a sphere (right).

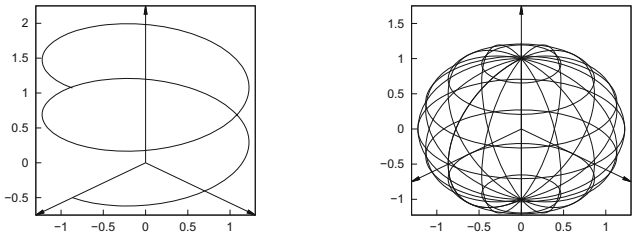

Fig. 7. Isometric axonometric projection of a circular helix (left) as well as the u-parameter and v-parameter curves of a sphere (right).

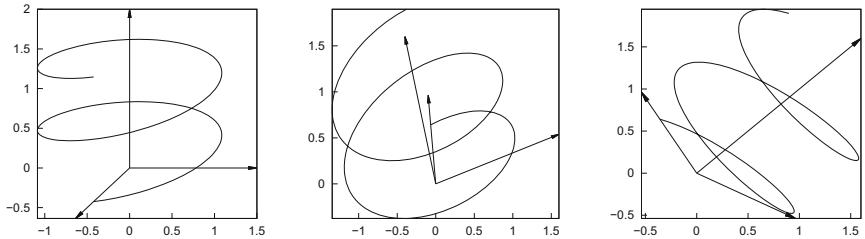

Fig. 8. Simulation of some rotations of the projection of the curve $\alpha(t) = (\cos(t), \sin(t), \frac{t}{8})$. The center and right figures exemplify trimetric axonometric projections.

However, all projections obtained lack movement in $\mathbb{R}^3$ but it is possible to simulate such movement using the product

$$R = R_z(\gamma)\, R_y(\beta)\, R_x(\alpha)\,,$$

which serves to represent a rotation around the z, y and x axes. Figure 8 was obtained after applying rotations to the coordinate functions of the helix

$$\alpha(t) = \left(\cos(t), \sin(t), \frac{t}{8}\right),\ 0 < t < 4\pi,$$

starting from the dimetric axonometric projection.

2.4 Axonometric Projections from $\mathbb{R}^4$ to $\mathbb{R}^3$

Pohlke's theorem was enunciated for projections of $\mathbb{R}^3$ on $\mathbb{R}^2$; however, it is possible to expand it naturally for projections of $\mathbb{R}^4$ on $\mathbb{R}^3$ [6,14,22]. For example, taking into account that the projection of the OX axis of Fig. 9 (right) is on the prolongation of the diagonal of a square (Fig. 9, left) it is possible to extend this idea to project the $OXYZW$ system on the $\bar{O}\bar{X}\bar{Y}\bar{Z}\bar{W}$ system, choosing the axes as shown in Fig. 9 (right). Note that the projection of the OX axis is chosen such that it rests on the prolongation of the diagonal of a cube, in addition

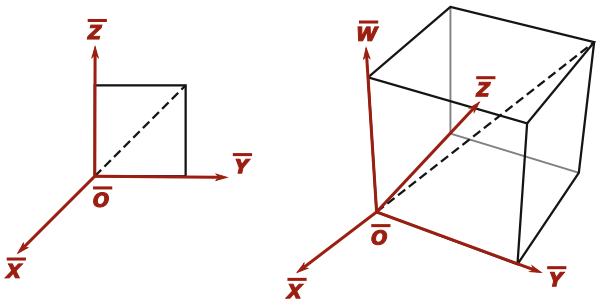

Fig. 9. The most common 2D axonometric projection (left) and a simile 3D axonometric projection (right).

$\sphericalangle\ \bar{Y}\bar{Z} = \sphericalangle\ \bar{Y}\bar{W} = \sphericalangle\ \bar{Z}\bar{W} = 90°$ and $\sphericalangle\ \bar{X}\bar{Y} = \sphericalangle\ \bar{X}\bar{Z} = \sphericalangle\ \bar{X}\bar{W} \approx 125°15'51.803''$. To find the analytical expressions of this projection, the coordinate system generated by $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}\}$, with axes: OX, OY, OZ, OW, will be projected on the coordinate system generated by $\{O'; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, with axes: $O'X'$, $O'Y'$, $O'Z'$.

Let's define the projection $\mathbf{p} : \mathbb{R}^4 \to \mathbb{R}^3$ (Fig. 10) for which it is true that

$$\begin{aligned}
&\mathbf{p}(\boldsymbol{i}) = -\frac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \\
&\mathbf{p}(\boldsymbol{j}) = \boldsymbol{e}_1 = (1,0,0), \\
&\mathbf{p}(\boldsymbol{k}) = \boldsymbol{e}_2 = (0,1,0), \\
&\mathbf{p}(\boldsymbol{l}) = \boldsymbol{e}_3 = (0,0,1).
\end{aligned}$$

Let $P(x, y, z, w)$ be an arbitrary point of $\mathbb{R}^4$ then

$$\mathbf{p}(P) = \mathbf{p}(x, y, z, w) = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \tag{4}$$

Equation (4) indicates that the point $P = (x, y, z, w)$ projects onto the point $P' = \left(-\frac{x}{\sqrt{3}} + y, -\frac{x}{\sqrt{3}} + z, -\frac{x}{\sqrt{3}} + w\right)$.

Another example is obtained extend the projection defined in (3). Figure 11 (left) shows that the origin of the $\bar{O}\bar{X}\bar{Y}\bar{Z}$ system is located on the centroid of an equilateral triangle and the axes start from the centroid towards the vertices of the triangle. In the same way, it is possible to locate the origin of the $\bar{O}\bar{X}\bar{Y}\bar{Z}\bar{W}$ system on the centroid of an equilateral tetrahedron and the axes that start from the centroid towards the vertices of the tetrahedron (Fig. 11, right). In this way the projection

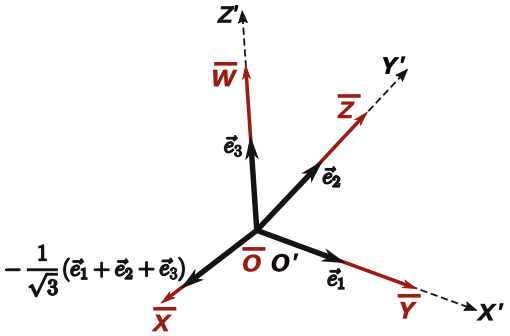

Fig. 10. Scheme to obtain the analytical expressions of a 3D axonometric projection.

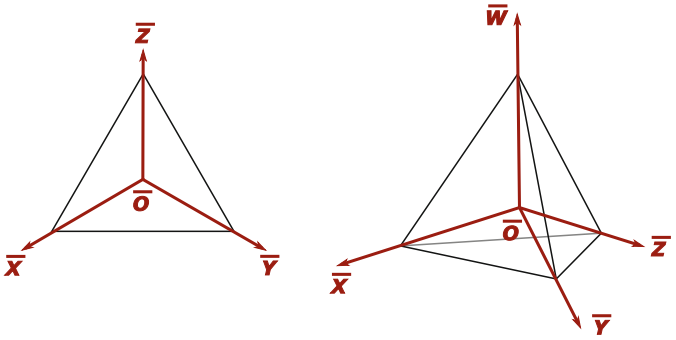

Fig. 11. Another common 2D axonometric projection (left) and a simile 3D axonometric projection (right).

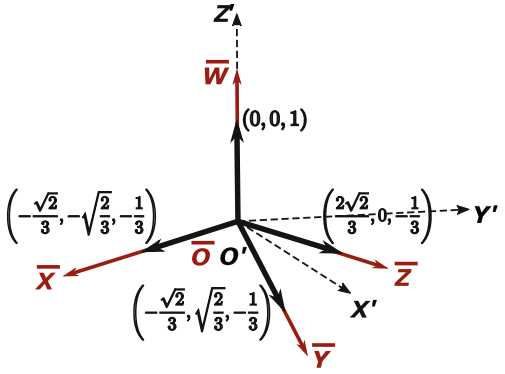

Fig. 12. Scheme to obtain the analytical expressions of another 3D axonometric projection.

$$\mathbf{p}(x,y,z,w) = \begin{pmatrix} -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \tag{5}$$

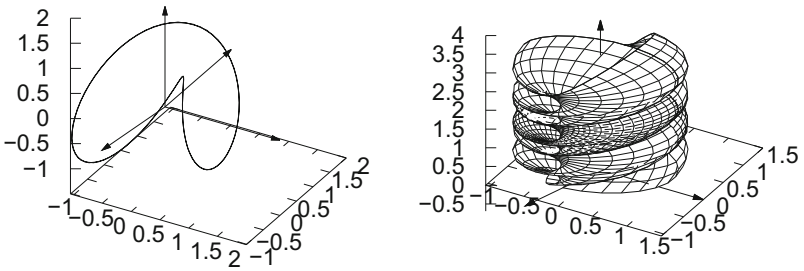

Fig. 13. Projection of the curve β (left) as well as the surface S (right) according to (4).

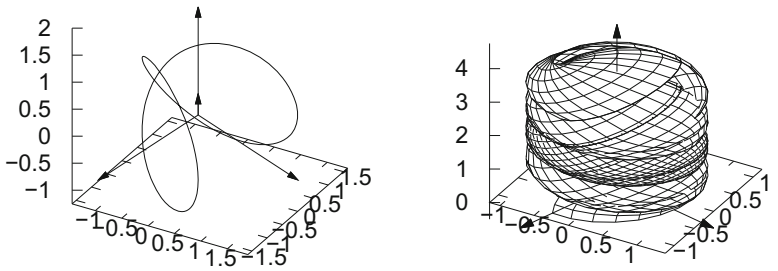

Fig. 14. Projection of the curve β (left) as well as the surface S (right) according to (5).

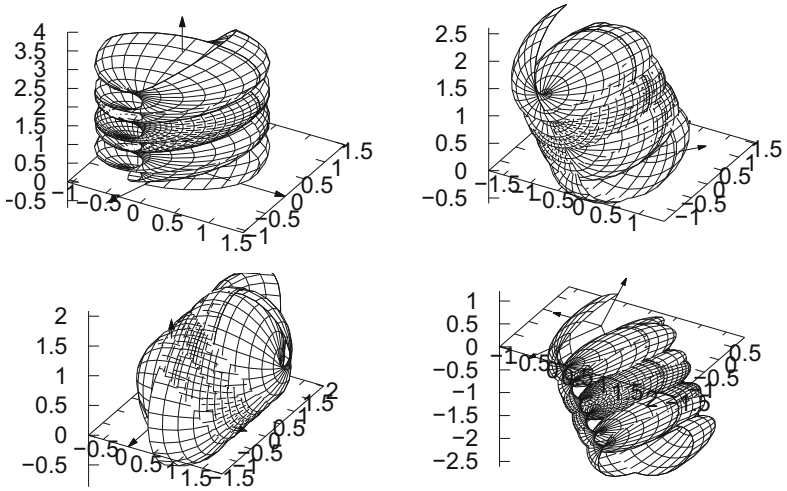

Fig. 15. Simulation of some rotations of the projection of the surface $S(u,v) = \left(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v), \frac{u+v}{8}\right)$.

is obtained (Fig. 12). Here the angles formed by the axes, taken two by two, measure approximately $161°33'54.1842''$.

Figure 13 shows the graphs of the curve

$$\beta(t) = (\cos(t), \sin(t), \cos(2t), \sin(2t)), \quad 0 < t < 2\pi,$$

(left) and of the surface

$$S(u,v) = \left(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v), \frac{u+v}{8}\right), \quad 0 < u, v < 4\pi,$$

(right) projected according to (4), while Fig. 14 shows the same graphs but projected according to (5). Note that, in the case of the projection according to (4), expanding the indications of (1), $v_x = \frac{3}{5}$, $v_y = v_z = v_w = 1$; while in the case of the projection according to (5), $v_x = v_y = v_z = v_w = 1$.

The analytical expressions of a projection $\mathbf{p}$ allow to obtain projections of mathematical objects from $\mathbb{R}^4$ to $\mathbb{R}^3$, according to the choice of $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ axes. However, all projections obtained lack movement in $\mathbb{R}^4$ but it is possible to simulate such movement using the product

$$R = R_{zw}(\psi)\, R_{yw}(\varphi)\, R_{yz}(\delta)\, R_{xw}(\gamma)\, R_{xz}(\beta)\, R_{xy}(\alpha),$$

which serves to represent a rotation around the zw, yw, yz, xw, xz and xy planes. Figure 15 was obtained after applying rotations to the coordinate functions of the surface $S(u,v) = \left(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v), \frac{u+v}{8}\right), 0 < u, v < 4\pi$.

3 The Package `R4projections`: Some Illustrative Examples

This section describes some examples of the application of this package. First, the package is loaded:

```
load(R4projections);
```

The two main functions incorporated in this package are:

```
draw4d, complexplot, plot4d.
```

With the `draw4d` function it is possible to visualize the plotting of the projections of points, lines, vectors, explicit functions, parametric curves, parametric surfaces and complex surfaces.

Next, a random list of points is plotted and then they are linked by a line.

```
p:create_list(map(random,[10,15,20,10]),k,1,10)$
draw4d( points(p) );
```

See Fig. 16 (left).

```
draw4d( points_joined=true,points(p) );
```

See Fig. 16 *(right).*

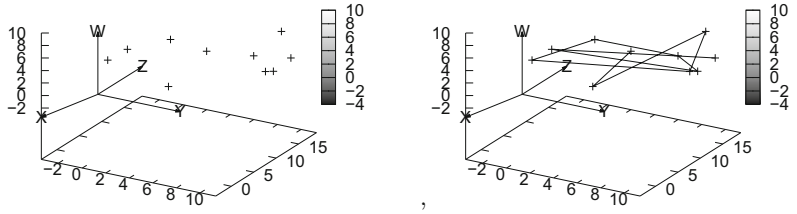

Fig. 16. Projections of points and lines from $\mathbb{R}^4$.

The following code allows obtaining the plotting of a set of vectors.

```
draw4d(  head_length=0.03,head_angle=10,
    vector([0,1/2,0,sqrt(3)/2],[-1/2,0,sqrt(3)/2,0]),
    vector([-1,0,0,0],[-1/2,0,0,-sqrt(3)/2]),
    vector([0,1/2,-sqrt(3)/2,0],[1,0,0,0])  )$
```

See Fig. 17.

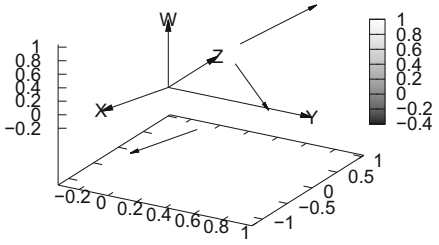

Fig. 17. Projections of vectors from $\mathbb{R}^4$.

To plotting one or more curves we would have to type the following code.

```
draw4d(  proportional_axes=xy,xyplane=-1.25,nticks=100,
    parametric(cos(t-1)*cos(t)^2,cos(t-1)*cos(t)*sin(t),
       cos(t-1)*sin(t),-sin(t-1),t,0,2*%pi)  )$
```

See Fig. 18 *(left).*

```
draw4d(  proportional_axes=xy,xyplane=-1.25,nticks=100,
    parametric(cos(t-1)*cos(t)^2,cos(t-1)*cos(t)*sin(t),
       cos(t-1)*sin(t),-sin(t-1),t,0,2*%pi),
    parametric(cos(t)^2*cos(2*t),cos(t)*sin(t)*cos(2*t),
       cos(t)*sin(2*t),sin(t),t,0,2*%pi)  )$
```

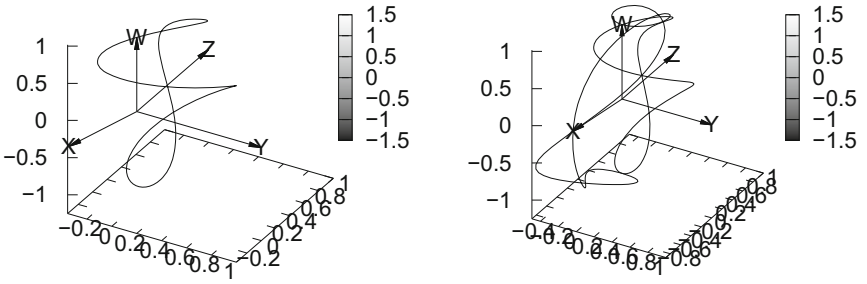

Fig. 18. Projections of curves from $\mathbb{R}^4$.

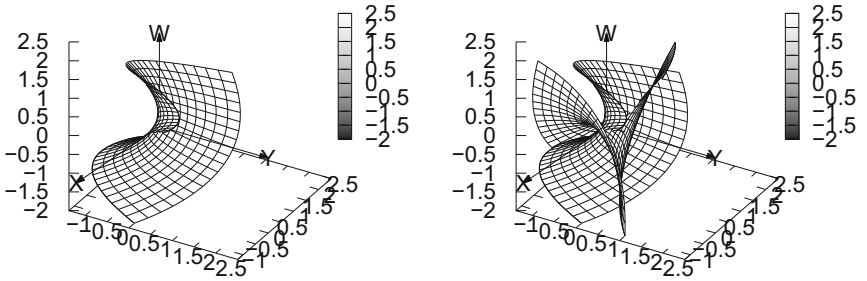

Fig. 19. Projections of surfaces from $\mathbb{R}^4$.

See Fig. 18 *(right).*

The `parametric_surface` function affected by the `draw4d` function allows to plotting one or more surfaces given in parametric form.

```
draw4d(  proportional_axes=xy,xyplane=-1.25,nticks=100,
    parametric_surface(u+v,u^2-v^2,u*v,u-v,u,-1,1,v,-1,1)  )$
```

See Fig. 19 *(left).*

```
draw4d(  proportional_axes=xy,xyplane=-1.25,nticks=100,
    parametric_surface(u+v,u^2-v^2,u*v,u-v,u,-1,1,v,-1,1),
    parametric_surface(u^2-v^2,u,v,2*u*v,u,-1,1,v,-1,1)  )$
```

See Fig. 19 *(right).*

Thanks to the isomorphism that exists between $\mathbb{C}$ and $\mathbb{R}^2$, as well as between $\mathbb{C}^2$ and $\mathbb{R}^4$, it is possible to obtain the graphs of complex functions such as $F(z) = z^2$ and $F(z) = \Gamma(z)$.

```
complexplot( z^2,[z,-1-%i,1+%i],[plot_format,openmath] )$
```

See Fig. 20 *(left).*

```
complexplot( gamma(z),[z,-3-3*%i,3+3*%i],[plot_format,openmath] )$
```

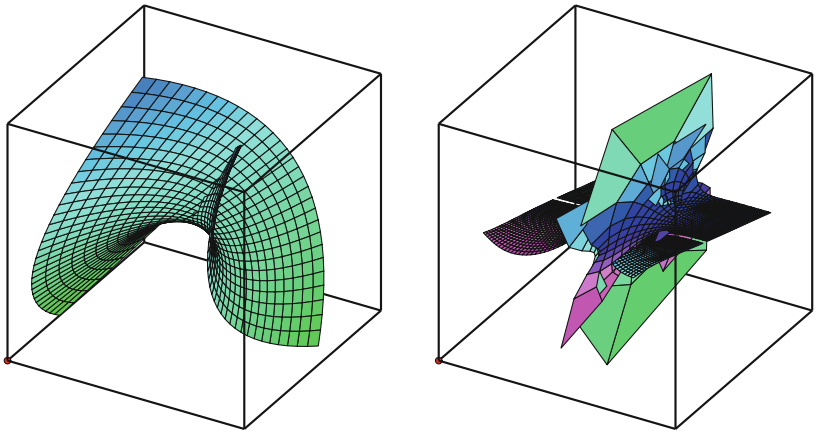

Fig. 20. Projections of complexes surfaces from $\mathbb{C}^2$.

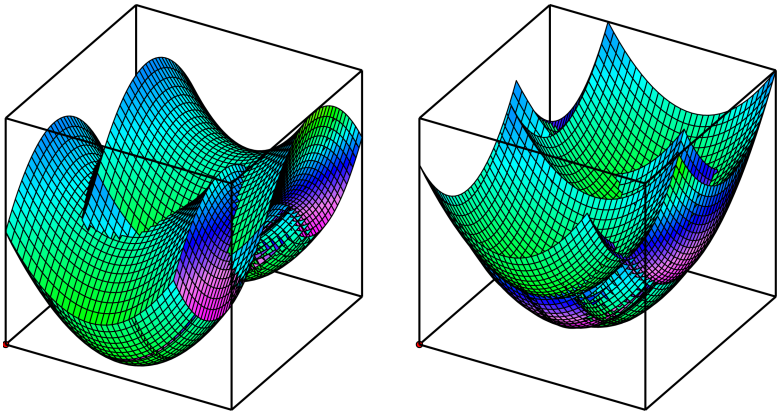

Fig. 21. Projections of solids from $\mathbb{R}^4$.

See Fig. 20 *(right).*

Finally the plotting of two solids are shown (Fig. 21).

```
plot4d(
      [u,v,w,u^2+v^2+w^2],[u,-1,1],[v,-1,1],[w,-1,1],
       [plot_format,openmath]
)$
```

See Fig. 20 *(left).*

```
plot4d(
      [u,v,w,u^2+v^2-w^2],[u,-1,1],[v,-1,1],[w,-1,1],
       [plot_format,openmath]
)$
```

See Fig. 20 *(right).*

4 Conclusions

In this paper, axonometric projections from $\mathbb{R}^4$ to $\mathbb{R}^3$ are constructed to obtain projections of objects immersed in four-dimensional space on a 3D hyperplane. These constructions are made as an extension of the construction of the axonometric projections from $\mathbb{R}^3$ to $\mathbb{R}^2$, which are based on the Pohlke theorem. Here is the natural extent of two of the most common projections from $\mathbb{R}^3$ to $\mathbb{R}^2$. Taking as a reference the diagonal of a square, for one, and the centroid of an equilateral triangle, for the other, it is possible to extend them two projections from $\mathbb{R}^4$ to $\mathbb{R}^3$ for which the diagonal of a cube and the centroid of a tetrahedron, respectively. Of course, it is possible to build countless of these projections. To simulate rotations in $\mathbb{R}^4$, in the same way as for $\mathbb{R}^3$, the six matrices associated with rotations around the six coordinate planes are used. In addition, the new *Maxima* package is described: `R4projections`. This package incorporates functions to plotting: points, lines, vectors, curves, surfaces, complex surfaces and solids; immersed in $\mathbb{R}^4$. Several illustrative examples, aimed to show the good performance of the package, are also given.

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