Calculation of the Differential Geometry Properties of Implicit Parametric Surfaces Intersection

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Abstract. Generally, to calculate the Frenet-Serret apparatus of a curve, it is necessary to have a parameterization of it; but when it is difficult to obtain a parameterization of the curve, as is the case of the curves obtained by the intersection of two implicit parametric surfaces, it is necessary to develop new methods that make it possible to know the geometric properties of said curve. This paper describes a new Mathematica package, Frenet, with the objective of calculating the properties of the differential geometry of a curve obtained by the intersection of two implicit parametric surfaces. The presented package allows us to visualize the Frenet-Serret mobile trihedron, to know the curvature and torsion at a given point of the curve obtained by the intersection of two implicit parametric surfaces. Package performance is discussed using several illustrative examples. Provide the user with an important tool for visualization and teaching.

Keywords: Frenet-Serret apparatus · Intersection of surfaces · Geometric-differential properties of curves

1 Introduction

In geometry, the study of the differential geometry properties of curves is essential. Generally, to calculate the Frenet-Serret apparatus of a curve, it is necessary to have a parameterization of it [2–4]; but when it is difficult to obtain a parameterization of the curve, as is the case of the curves obtained by the intersection of two implicit parametric surfaces, it is necessary to develop new methods that allow knowing the geometric properties of the curve; methods that have been studied in [1] but this calculation becomes a very cumbersome task due to the amount of mathematical operations that must be carried out for this reason this paper describes a new Mathematica package, Frenet, with objective of calculating the properties of the differential geometry of a curve obtained by the intersection of two implicit parametric surfaces. Enabling the calculation of the Frenet-Serret apparatus of a curve without having a parameterization of it. The outputs obtained

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are consistent with Mathematica notation and results. Package performance is discussed using several illustrative examples. The presented package allows us to visualize the Frenet-Serret mobile trihedron, to know the curvature and torsion at a given point of the curve obtained by the intersection of two implicit parametric surfaces, providing the user with a very useful tool for teaching and visualization. The paper is organized as follows: In Sect. 2, the formulas necessary to calculate the properties of the differential geometry of a curve obtained by the intersection of two implicit parametric surfaces are reviewed. In the Sect. 3 introduces the new Mathematica package, Frenet, and describes the implemented commands. Package performance is also analyzed using three explanatory examples. We finish Sect. 4 with the main conclusions of this paper.

2 Mathematical Preliminaries

2.1 Curves

Let $\beta : I \subset \mathbb{R} \to \mathbb{R}^3$ an arbitrary curve with arc-length parametrization, then from the elementary differential geometry, we have

$$\beta'(s) = t$$

$$\beta''(s) = k = \kappa n$$

where t is the unit tangent vector, n is the unit principal normal vector and k is the curvature vector. The curvature is given by

$$k = \sqrt{\kappa \cdot \kappa} = \sqrt{\beta'' \cdot \beta''}$$

the binormal unit vector b is defined by $b = t \times n$. The vectors $\{t, n, b\}$ are called collectively the Frenet- Serret frame, the Frenet–Serret formulas along β are given by

$$t'(s) = \kappa n$$

$$n'(s) = -\kappa t + \tau b$$

$$b'(s) = -\tau n$$

where τ is the torsion of the curve β.

2.2 Implicit Surface Representation

A surface often arises as the locus of a point P which satisfies some restriction, as a consequence of which the coordinates x, y, z of P satisfy a relation of the form

This is called the implicit or constraint equation of the surface, assume that $f(x, y, z) = 0$ is a regular implicit surfaces. In other words $\nabla f \neq 0$, where $\nabla f = (f_1, f_2, f_3)$ is the gradient vector of the surface f, where $f_1 = \frac{\partial f}{\partial x}$, $f_2 = \frac{\partial f}{\partial y}$ and $f_3 = \frac{\partial f}{\partial z}$ denote to partial derivatives of the surface f, then the unit surface normal vector field of the surface f is given by

$$n = \frac{\nabla f}{\|\nabla f\|}$$

2.3 Parametric Surface Representation

A parametric surface in the Euclidean $\mathbb{R}^3$ is defined by a parametric equation with two parameters. Parametric representation is the most general way to specify a surface. In general, if we take the real parameters u and v, then the surfaces can be defined by the vector-value function, $r = r(u, v)$, where $r(u, v) = r(r_1(u, v), r_2(u, v), r_3(u, v))$ and $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$.

2.4 Properties of Parametric Surfaces

Given a parametric surface of the form $r(u, v) = r(r_1(u, v), r_2(u, v), r_3(u, v))$, supposing that $r(u, v)$ is a regular parametric surface. That is to say $r_u \times r_v \neq 0$, where $r_u = \frac{\partial r}{\partial u}$ and $r_v = \frac{\partial r}{\partial v}$ denote to partial derivatives of the surface r.
The unit vector normal n at any point on a parametric surfaces is obtained as

$$n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

the first fundamental form given a parametric surface $r(u, v)$, we define the first fundamental form coefficients $E = r_u \cdot r_u$, $F = r_u \cdot r_v$ and $G = r_v \cdot r_v$, then the first fundamental form I of the surface is the quadratic expression defined as,

$$I = Edu^2 + 2Fdudv + Gdv^2$$

the second fundamental form given a parametric surface $r(u, v)$ and its normal vector n, we define the second fundamental form coefficients $L = r_{uu} \cdot n$, $M = r_{uv} \cdot n$ and $N = r_{vv} \cdot n$ then the second fundamental form of the surface is the quadratic expression defined as

$$II = Ldu^2 + 2Mdudv + Ndv^2$$

3 Transversal Intersection of Parametric–Implicit Surfaces

Let $r = r(u, v)$ and $f(x, y, z) = 0$ two surfaces regular with unit normal vectors given by

$$n_1 = \frac{r_u \times r_v}{|r_u \times r_v|}$$

$$n_2 = \frac{\nabla f}{\|\nabla f\|}$$

The intersection curve of these surfaces can be seen as a curve of both surfaces as

$$\beta(s) = (x(s), y(s), z(s)), f(x, y, z) = 0$$

$$\beta(s) = r(u(s), v(s)), c_1 < u < c_2, c_3 < v < c_4$$

then you have

$$x(s) = r_1(u(s), v(s))$$
$$y(s) = r_2(u(s), v(s))$$
$$z(s) = r_3(u(s), v(s))$$

where

$$r(u(s), v(s)) = (r_1(u(s), v(s)), r_2(u(s), v(s)), r_3(u(s), v(s)))$$

The surface f can be expressed as

$$h(u, v) = f(r_1, r_2, r_3) = 0$$

Thus the intersection curve is given by

$$\beta(s) = r(u(s), v(s)), h(u, v) = 0, c_1 < u < c_2, c_3 < v < c_4$$

We can easily find the first derivative of the intersection curve

$$\beta(s) = r_u u' + r_v v' \tag{1}$$

the tangent vector of the transversal intersection curve $\beta(s)$ lies on the tangent planes of both surfaces. Therefore it can be obtained as the cross product of the unit normal vectors of the two surfaces at P as

$$t = \frac{n_1 \times n_2}{|n_1 \times n_2|} \tag{2}$$

the surface curvature vector $r(u, v)$ is given by

$$\kappa_1 = L(u')^2 + 2Mu'v' + N(v')^2 \tag{3}$$

since we know the unit tangent vector of the intersection curve from Eq. (2), we can find u' and v' by taking the dot product on both hand sides of Eq. (1) with r_u and r_v , which leads to a linear system

$$Eu' + Fv' = r_u.t$$

$$Fu' + Gv' = r_v.t$$

where E, F, G are the first fundamental form coefficients.
Thus

$$u' = \frac{(r_u \cdot t)G - (r_v \cdot t)F}{EG - F^2}$$

$$v' = \frac{(r_v \cdot t)E - (r_u \cdot t)F}{EG - F^2}$$

similarly

$$\beta'(s) = t = (x'(s), y'(s), z'(s))$$

$$\beta''(s) = k = \kappa n = (x''(s), y''(s), z''(s))$$

where x', y', z' are the three components of t given by Eq. (2) and x'', y'', z'' are the three components of k.

We can calculate the normal curvature of the implicit surface using the equation

$$\kappa_2 = -\frac{f_{xx}(x')^2 + f_{yy}(y')^2 + f_{zz}(z')^2 + 2\left(f_{xy}x'y' + f_{yz}y'z' + f_{xz}x'z'\right)}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$$

consequently, the curvature vector of the intersection curve $\beta(s)$ at P can be calculated as follows:

$$k = \frac{\kappa_1 - \kappa_2 \cos\theta}{\sin^2\theta} n_1 + \frac{\kappa_2 - \kappa_1 \cos\theta}{\sin^2\theta} n_2 \tag{4}$$

the curvature of the intersection curve $\beta(s)$ at P can be calculated using Eq. (4) as follows

$$\kappa = \sqrt{k \cdot k} = \frac{1}{|\sin\theta|}\sqrt{\kappa_1^2 + \kappa_2^2 - 2\kappa_1^2\kappa_2^2\cos\theta}$$

the unit normal vector and unit binormal vector of the intersection curve $\beta(s)$ given as

$$n = \frac{k}{\kappa}, b = t \times n$$

the torsion of the intersection curve $\beta(s)$ is obtained by

$$\tau = \frac{1}{\kappa \sin^2\theta}\left\{[\lambda_1 - \lambda_2\cos\theta](b \cdot n_1) + [\lambda_2 - \lambda_1\cos\theta](b \cdot n_2)\right\}$$

where

$$\lambda_1 = 3\left[Lu'u'' + M(u''v' + u'v'') + Nv'v''\right] + r_{uuu} \cdot n(u')^3 + 3r_{uuv}n(u')^2v'2 + 3r_{uvv} \cdot nu'(v')^2 + r_{vvv} \cdot n(v')^3$$

the values of u'', v'' are obtained by solving the following system of equations

$$Eu'' + Fv'' = k \cdot r_u - \frac{E_u}{2}(u')^2 - E_v u'v' - \left(F_v - \frac{G_u}{2}\right)(v')^2$$

$$Fu'' + Gv'' = k \cdot r_v - \left(F_u - \frac{E_v}{2}\right)(u')^2 - G_u u'v' - \frac{G_v}{2}(v')^2$$

similarly

$$\lambda_2 = -\frac{F_1 + F_2 + F_3}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$$

the values of F_1, F_2, F_3 can be calculated by

$$F_1 = f_{xxx}(x')^3 + f_{yyy}(y')^3 + f_{zzz}(z')^3$$

$$F_2 = 3\Big[f_{xxy}(x')^2y' + f_{xxz}(x')^2z' + f_{xyy}x'(y')^2 + f_{yyz}(y')^2z' + f_{xzz}x'(z')^2 f_{yzz}y'(z')^2 + 2f_{xyz}x'y'z'\Big]$$

$$F_3 = 3\left[f_{xx}x'x'' + f_{yy}y'y'' + f_{zz}z'z'' + f_{xy}\left(x''y' + x'y''\right) + f_{yz}\left(y''z' + y'z''\right) + f_{xz}\left(x''z' + x'z''\right)\right]$$

4 The Package Frenet: Some Illustrative Examples

In this section we describe the use of the Frenet package, the package works with Mathematica v.11.2 and later versions. Various examples will be used for an introduction to the specific features of the package. First of all, we load the package:

$$<< Frenet.m$$

4.1 Example 01

The first example is given by the curve obtained by the intersection of the implicit surface $f(x, y, z) = z - y^2 - 2 = 0$ and the parametric surface $r(u, v) = (u, uv, v)$.

With the *Frenet* command it is possible to calculate the equations of the Frenet-Serret apparatus at a generic point on the curve.

$$Frenet[z - y^2 - 2, u, u * v, v, x, y, z, u, v]//Simplify$$

Frenet returns the equations of the tangent vector, normal vector, binormal vector as well as curvature and torsion.

$$\left\{\left\{\frac{2uy-1}{\sqrt{u^2+v^2+1}\sqrt{4y^2+1}\sqrt{\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}}}, -\frac{v}{\sqrt{u^2+v^2+1}\sqrt{4y^2+1}\sqrt{\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}}}, -\frac{2vy}{\sqrt{u^2+v^2+1}\sqrt{4y^2+1}\sqrt{\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}}}\right\},\right.$$

$$\left\{-\frac{v^2\left(2y\left(4y^2+v+1\right)+u\left(v-4\left(4y^4+y^2\right)\right)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^2\sqrt{\frac{v^2\left(v^4+\left(u^2+1\right)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2\left(4y^2+1\right)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}}},\right.$$

$$\frac{v\left(8u^2y^3-8uy^2-2\left(v^3+u^2v-1\right)y+uv\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^2\sqrt{\frac{v^2\left(v^4+\left(u^2+1\right)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2\left(4y^2+1\right)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}}},$$

$$\left.\left.\frac{v\left(v^3-2uyv+v+4y^2(1-2uy)^2\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^2\sqrt{\frac{v^2\left(v^4+\left(u^2+1\right)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2\left(4y^2+1\right)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}}}\right\},\right.$$

$$\left\{-\frac{v^3}{(u^2+v^2+1)^{3/2}(4y^2+1)^{3/2}\left(\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}\right)^{3/2}\sqrt{\frac{v^2\left(v^4+(u^2+1)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2(4y^2+1)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}}},\right.$$

$$\frac{v\left(4(1-2uy)y^2+v\right)}{(u^2+v^2+1)^{3/2}(4y^2+1)^{3/2}\left(\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}\right)^{3/2}\sqrt{\frac{v^2\left(v^4+(u^2+1)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2(4y^2+1)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}}},$$

$$\left.-\frac{v\left(2y+u\left(v-4y^2\right)\right)}{(u^2+v^2+1)^{3/2}(4y^2+1)^{3/2}\left(\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}\right)^{3/2}\sqrt{\frac{v^2\left(v^4+(u^2+1)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2(4y^2+1)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}}}\right\},$$

$$2\sqrt{\frac{v^2\left(v^4+(u^2+1)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2(4y^2+1)\right)}{\left((4y^2+1)v^2+(1-2uy)^2\right)^3}},$$

$$\Big(3v\left(-2v\left(u^2+v^2+1\right)^{3/2}(u(u+v)+1)y^2\left(4y^2+1\right)^{3/2}(2v)\right.$$

$$\frac{2uy-1}{\sqrt{u^2+v^2+1}\sqrt{4y^2+1}\sqrt{\frac{(4y^2+1)v^2+(1-2uy)^2}{(u^2+v^2+1)(4y^2+1)}}}\left(\frac{\left(4y^2+1\right)v^2+(1-2uy)^2}{\left(u^2+v^2+1\right)\left(4y^2+1\right)}\right)^{3/2}$$

$$-4\left(u^2+1\right)y^2(2uy-1)^3+4uv^6y^2(4uy-1)+v^5(4uy-1)\left(4\left(u^2+1\right)y^2-2uy+1\right)-2v^2y(2uy-1)\left(-4yu^3+2u^2+4y^2u+u+8y^3+2y\right)+2uv^4y\left(32u^2y^4-24uy^3+4\left(2u\left(u^2+u+1\right)+1\right)y^2-2(3u(u+1)+1)y+2u+1\right)+v^3(-32u^2\left(2u^2+1\right)y^5$$

$$+16u\left(5u^2+1\right)y^4+8u\left(u\left(2u^3+4u-5\right)+2\right)y^3-4\left(u\left(5u^3+7u-2\right)+1\right)y^2+2u\left(5u^2+7\right)y-u^2-2\Big)+v(1-2uy)^2\left(u\left(-16uy^3\right.\right.$$

$$\left.\left.+8y^2+4\left(u^2+1\right)y-u\right)-1\right)\Big\}\left(v^2+1\right)\left(u(u+v)+1\right)\left(4y^2+1\right)v^2+(1-2uy)^2$$

$$\left(v^4+\left(u^2+1\right)v^2-4y(u+2y)(2uy-1)v+4y^2(1-2uy)^2\left(4y^2+1\right)\right)\Big\}\Big\}$$

The following sentences allow us to obtain the Frenet-Serret apparatus of the curve at one point $(0, 0, 2)$.

$\{t, b, n, k, \tau\} = \text{Frenet}\left[\left(-y^2+z-2, \{u, uv, v\}, \{x, y, z\}, \{u, v\}, \text{ParVals} \rightarrow \{0, 0, 2, 0, 2\}\right]\right.$ //Simplify

Frenet returns the tangent vector, normal vector, binormal vector as well as curvature and torsion at one point $(0, 0, 2)$.

$$\left\{\left\{-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right\}, \{0, 0, 1\}, \left\{-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\}, 8/5, -\frac{3}{5}\right\}$$

The following sentences allow us to obtain the graphs of the Frenet-Serret apparatus at the point (0,0,2), the red vector represents the tangent vector, the green vector represents the normal vector and the celestial vector represents the binormal vector.

$pp = \{0, 0, 2\}; Show[ContourPlot3D[z - y^2 - 2 == 0, \{x, -1, 1\},$
$\{y, -2, 2\}, \{z, 1, 3\}, ContourStyle \rightarrow Blue],$
$ParametricPlot3D[\{u, uv, v\}, \{u, -1, 1\}, \{v, 1, 3\},$
$PlotStyle \rightarrow Opacity[0.75]], Graphics3D[\{\{AbsolutePointSize[8], Point[pp]\},$
$\{Red, Arrow[Tube[\{pp, pp + t\}]]\}, \{Green, Arrow[Tube[\{pp, pp + n\}]]\},$
$\{Cyan, Arrow[Tube[\{pp, pp + b\}]]\}\}]]$

See Fig. 1.

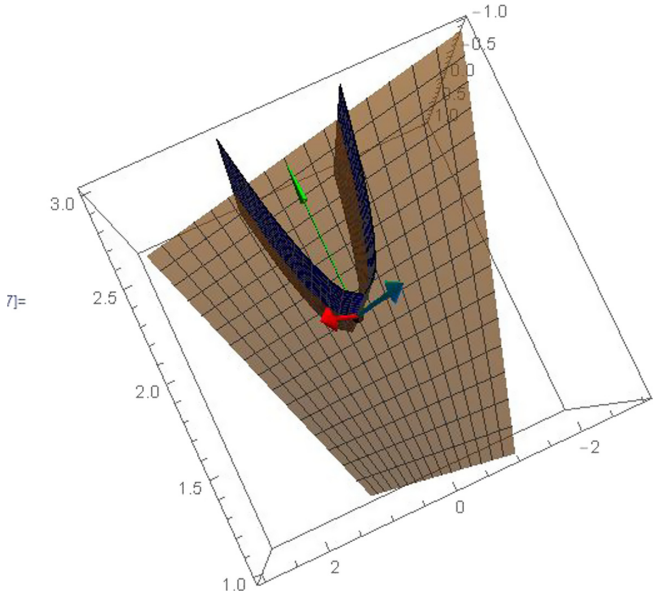

Fig. 1. Graphical visualization of the Frenet-Serret apparatus of the curve of intersection of the implicit $f(x,y,z) = z - y^2 - 2 = 0$ surface and the parametric $r(u,v) = (u, uv, v)$ surface at the point $(0,0,2)$ (Color figure online)

4.2 Example 02

The second example is given by the curve obtained by the intersection of the implicit surface $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$ and the parametric surface $\{\cos(u)\cos(v), \cos(v)\sin(u), \sin(v)\}$.

The following sentences allow us to obtain the Frenet-Serret apparatus at the point of the curve $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right\}$.

$$\{t, b, n, k, \tau\} = \text{Simplify}\left[\text{Frenet}\left(\left(x - \frac{1}{2}\right)^2 + y^2 - \frac{1}{4}, \{\cos(u)\cos(v),\right.\right.$$
$$\left.\left.\sin(u)\cos(v), \sin(v)\}, \{x, y, z\}, \{u, v\}, \text{ParVals} \to \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}, \frac{\pi}{4}\right\}\right)\right]$$

Frenet returns the tangent vector, the normal vector, the binormal vector, as well as the curvature and torsion.

$$\left\{\left\{-\sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{3}}\right\}, \left\{-\frac{1}{\sqrt{39}}, -2\sqrt{\frac{3}{13}}, -\sqrt{\frac{2}{39}}\right\}, \left\{\frac{2}{\sqrt{13}}, -\frac{1}{\sqrt{13}}, 2\sqrt{\frac{2}{13}}\right\},\right.$$
$$\left.\frac{2\sqrt{\frac{13}{3}}}{3}, \frac{6}{13}\sqrt{3}\left(-2\sqrt{\frac{2}{3}} + \sqrt{6}\right)\right\}$$

The following sentences allow us to obtain the graphs of the Frenet-Serret apparatus at the point $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\}$, the red vector represents the tangent vector, the green vector represents the normal vector and the celestial vector represents the binormal vector.

$$pp = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right\}; Show\left[ContourPlot3D\left[\left(x - \frac{1}{2}\right)^2 + y^2 == 1/4,\right.\right.$$
$$[\{\cos[u]\cos[v], \cos[v]\sin[u], \sin[v]\}, \{u, -Pi, Pi\}, \{v, -Pi, Pi\},$$
$$PlotStyle \rightarrow Opacity[0.5]], Graphics3D[\{\{AbsolutePointSize[8],$$
$$Point[pp]\}, \{\text{Red}, Arrow[Tube[\{pp, pp + t\}]]\}, \{Green, Arrow[Tube$$
$$\left.\left.[\{pp, pp + n\}]]\}, \{Cyan, Arrow[Tube[\{pp, pp + b\}]]\}\}], PlotRange \rightarrow All\right]$$

See Fig. 2.

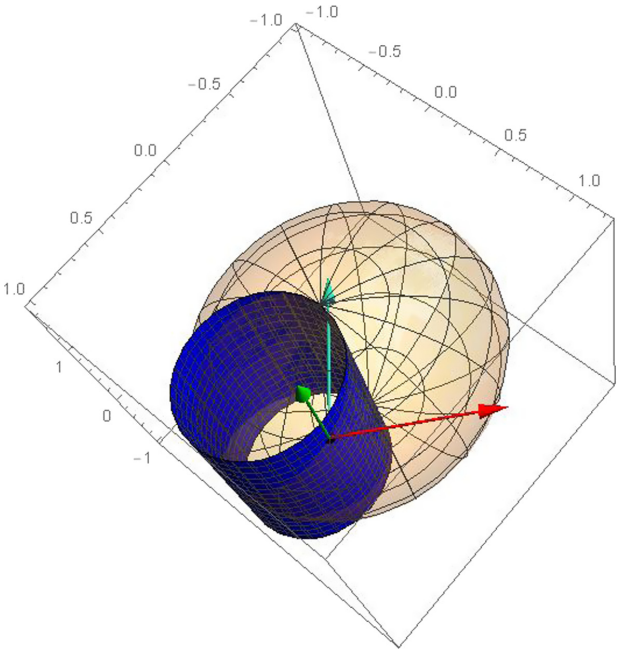

Fig. 2. Graphical visualization of the Frenet-Serret apparatus of the curve of intersection of the implicit $f(x, y, z) = (x - \frac{1}{2})^2 + y^2 - 1/4 = 0$ surface and the parametric $r(u, v) = \{\cos(u)\cos(v), \cos(v)\sin(u), \sin(v)\}$ surface at the point $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\}$ (Color figure online)

4.3 Example 03

The third example is given by the curve obtained by the intersection of the implicit surface $x^2+y^2 = 2$ and the parametric surface $\{u^3, v^3, uv\}$. The following sentences allow us to obtain the Frenet-Serret apparatus at the point of the $\{1, 1, 1\}$ curve.

$$\{t, b, n, k, \tau\} = \text{Frenet}\left[x^2 + y^2 - 2, \left\{u^3, v^3, \text{uv}\right\}, \{x, y, z\}, \{u, v\}, \text{ParVals} \rightarrow \{1, 1, 1, 1, 1\}\right] //FullSimplify$$

Frenet returns the tangent vector, the normal vector, the binormal vector, as well as the curvature and torsion.

$$\left\{\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}, \{0, 0, 1\}, \frac{1}{\sqrt{2}}, 0\right\}$$

The following sentences allow us to obtain the graphs of the Frenet-Serret apparatus at the point $\{1, 1, 1\}$, the red vector represents the tangent vector, the green vector represents the normal vector and the celestial vector represents the binormal vector.

$$pp = \{1, 1, 1\}; Show[ContourPlot3D[x^2 + y^2 == 2, \{x, -Pi, Pi\}, \{y, -Pi, Pi\}, \{z, -2, 2\}, ContourStyle \rightarrow Blue], ParametricPlot3D[\{u^3, v^3, uv\}, \{u, -1.3, 1.3\}, \{v, -1.3, 1.3\}, PlotStyle \rightarrow Opacity[0.5]], Graphics3D[\{\{AbsolutePointSize[8], Point[pp]\}, \{Red, Arrow[Tube[\{pp, pp + t\}]]\}, \{Green, Arrow[Tube[\{pp, pp + n\}]]\}, \{Cyan, Arrow[Tube[\{pp, pp + b\}]]\}\}], PlotRange \rightarrow All]$$

See Fig. 3.

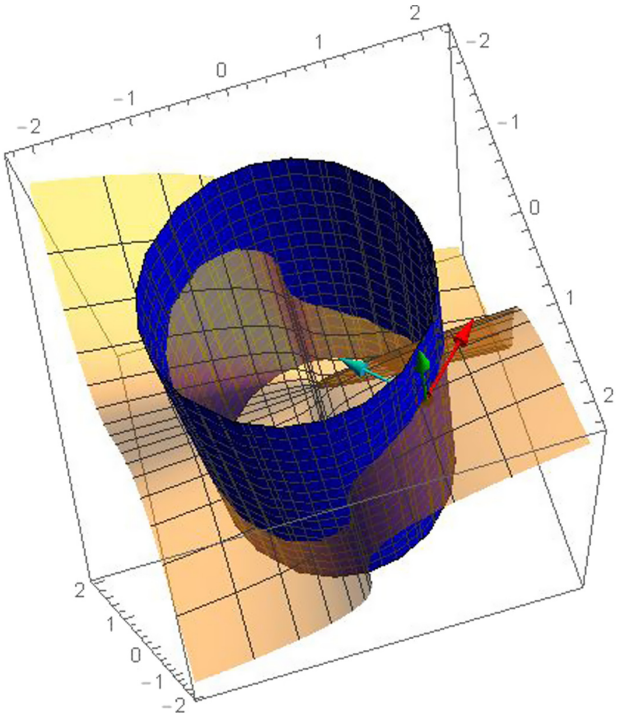

Fig. 3. Graphical visualization of the Frenet-Serret apparatus of the curve of intersection of the implicit $f(x, y, z) = x^2 + y^2 - 2 = 0$ surface and the parametric $r(u, v) = \{u^3, v^3, uv\}$ surface at the point $(1, 1, 1)$ (Color figure online)

5 Conclusions

This paper proposes a program implemented in Mathematica v.11.2 software to calculate the differential geometry properties of curves given by the intersection between two implicit parametric surfaces, based on the results obtained in [1,5] and as a continuation of [6] and [8] whose results coincide with those found in this paper. Demonstrating that it is possible to calculate the Frenet-Serret apparatus of a curve for which it is not necessary to know a parameterization, being of great help when performing operations that can often make said task cumbersome and also provide us with a very useful graphic representation of the problem. Package performance is discussed by means of some illustrative and interesting examples. All of the commands have been implemented in Mathematica v11.2 and are consistent with Mathematica notation and results [7,9,10]. The program is shorter and more efficient from my experience.

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