Generalized Invariants of Multiple Scattered Polarized Radiation at Arbitrary Non-symmetrical Optical Levels of a Uniform Slab

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Abstract. Polarimetric invariants of a uniform slab of a finite optical thickness, which were initially introduced at optical levels symmetrical to its middle in mirror directions, for downgoing and upgoing polarized radiation have been generalized for the case of virtual non-symmetrical optical levels τ_1 and τ_2. In the case of the vertical uniformity of an extended uniform slab, there deterministically exist invariant constructions different from and adequate to standard polarimetric invariants at symmetrical optical levels). These constructions express the properties of the local spatial angular symmetry of intensities of multiple scattered radiation internal fields in a more general analytical form of representation. For such new constructions of polarized radiation, which are called generalized polarimetric invariants, a basic boundary problem of the polarized radiation transfer theory has been formulated and linear Fredholm integral equations of the second kind and linear singular integral equations have been obtained. These generalized polarimetric invariants have proved to be effective for numerical modeling of intensities of multiple scattered polarized radiation in a uniform slab of an arbitrary optical thickness $\tau_0 \leq \infty$, with and without a reflecting bottom.

Keywords: Generalized polarimetric invariants · Non-symmetrical optical levels · Fredholm linear integral equations · Linear singular equations · Basic boundary value problems · Semi-group transformations · Singular eigenfunction

1 Introduction

The concept of polarized radiation field invariants was initially developed for downgoing and upgoing vision directions at optical levels that are mirror symmetrical relative to the middle of a uniform slab of finite optical thickness. Below it will be shown that symmetry restrictions can be eliminated, and generalized photometrical invariants can be introduced at virtual optical levels within any local layers in a given uniform slab. For that, we make required linear semigroup transformations of spatial shift and rotation relative to the virtual boundaries of local layers inside the initial uniform slab.

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2 The Basic Boundary Value Problem and Linear Fredholm Integral Equations of the Second Kind for Generalized Polarimetric Invariants

If we select an arbitrary local layer $[\tau_1 < \tau_2]$ in some initial uniform slab of a finite optical thickness $\tau_0 < \infty$, then we can eliminate the restriction specified in [1] regarding the symmetry of base optical levels τ and τ_0–τ. In this case, at its virtual boundaries τ_1 and τ_2 in the mirror directions of vision η and $-\eta$ at fixed values of the solar zenith angle ξ we can introduce new objects of the radiative transfer theory, namely generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$ of Fourier harmonics for matrix intensities of polarized radiation which are determined by the following relations [2]:

$$\begin{aligned} &\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0) = \hat{D}\hat{\hat{I}}^m(\tau_2, -\eta, \xi, \tau_0) \pm \hat{\hat{I}}^m(\tau_1, \eta, \xi, \tau_0), \\ &\eta \in [-1, 1] \cap \xi \in [0, 1], \quad 0 \le \tau_1 < \tau_2 \le \tau_0 < \infty, \quad m = \overline{0, M}. \end{aligned} \tag{1.1}$$

Azimuthal harmonics of matrix intensities $\hat{\hat{I}}^m(\tau_2, -\eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_1, \eta, \xi, \tau_0)$ in the downgoing ($\eta > 0$) and upgoing ($\eta < 0$) vision line directions are determined by special Fourier transformations [3], while $\hat{D}(1, 1, -1, -1)$ is a numerical diagonal matrix of mirror transformations of polarized radiation fields.

The matrix values $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$ introduced according to (1.1) possess the basic property of spatial angular invariance for mutual inversions of the virtual boundaries τ_1 and τ_2 of the selected local layers $[\tau_1, \tau_2]$ and for simultaneous substitution of vision line directions $\eta \leftrightarrow -\eta$:

$$\begin{aligned} &\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0) = \pm\hat{D}\hat{\hat{I}}^m(\tau_2, \tau_1 - \eta, \xi, \tau_0), \\ &\eta \in [-1, 1] \cap \xi \in [0, 1], \quad 0 \le \tau_1 < \tau_2 \le \tau_0 \le \infty, \quad m = \overline{0, M}. \end{aligned} \tag{1.2}$$

From relations (1.1)–(1.2) it follows that the new polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$ are a generalization of the earlier introduced analogous values $\hat{\hat{I}}^m_{\pm}(\tau, \eta, \xi, \tau_0)$ at the mirror symmetrical optical levels τ_1 and $\tau_2 - \tau$ of a uniform slab in "mirror" vision line directions η and $-\eta$ [1]. In that case, the azimuthal harmonics of generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$, unlike standard matrix values $\hat{\hat{I}}^m_{\pm}(\tau, \eta, \xi, \tau_0)$, exist outside the restrictions associated with the principle of mirror reflection (symmetry) and, as it will be demonstrated below, are determined by linear semi-group transformations of the sought for matrix intensities $\hat{\hat{I}}^m(\tau_1, \eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_2, -\eta, \xi, \tau_0)$ in relation to vertical spatial shifts of the virtual optical depths τ_1 and τ_2 with the simultaneous rotation of the vision lines $\eta \Leftrightarrow -\eta$.

Note also that exact functional relations and integral equations (including the basic boundary value problem), which are required for finding azimuthal harmonics of

generalized matrix polarimetric invariants $\hat{I}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$, are a result of applying two different approaches. The first one is connected with linear transformations of regular solutions for the initial boundary value problem in the polarize radiation transfer theory [2] and subsequent application of respective representations of generalized polarimetric invariants $\hat{I}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$ within arbitrary selected local layers $\tau \in [\tau_1, \tau_2]$. The second approach is based on the results obtained earlier [4] while solving the said boundary value problem according to the Case method [5] and subsequently used for considering problems of multiple polarized radiation scattering in a uniform semi-infinite slab ($\tau_0 = \infty$) [6]. The following strict linear Fredholm integral equations of the second kind and linear singular integral equations for the generalized polarimetric invariants $\hat{I}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$ can be utilized in the case of uniform media of either a finite ($\tau_0 < \infty$) or infinite ($\tau_0 = \infty$) optical thickness.

To determine the azimuthal harmonics of polarized radiation matrix intensities $\hat{I}^m(\tau_1, \eta, \xi, \tau_0)$ and $\hat{I}^m(\tau_2, -\eta, \xi, \tau_0)$ at non-symmetrical optical levels τ_1 and τ_2, we initially apply the first approach. Following this approach, let's consider regular solutions $\hat{I}^m(\tau, \eta, \xi, \tau_0), \eta \in [-1, 1] \cap \xi \in [0, 1] \cap \tau \in [0, \tau_0]$, $m = \overline{0, M}$ [1, 2], from the perspective of their linear semi-group transformations associated with the shift of

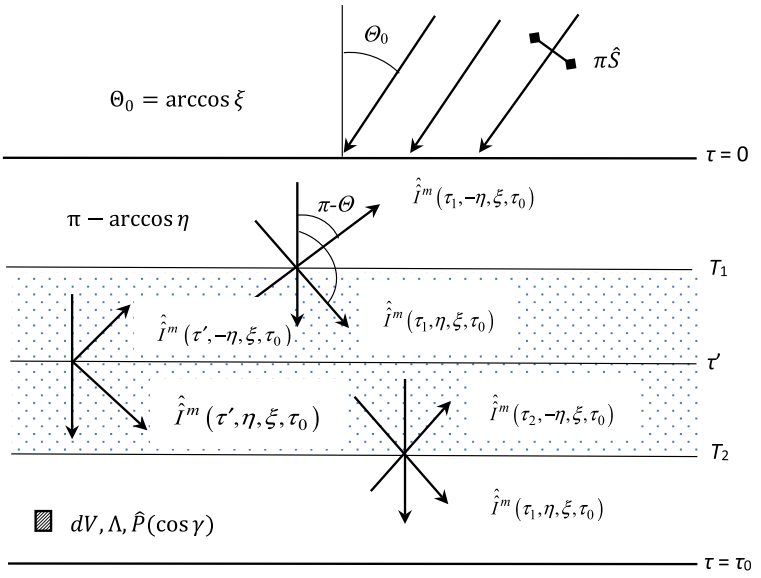


Fig. 1. Azimuthal harmonics of polarized radiation matrix intensities $\hat{I}^m(\tau_1, \eta, \xi, \tau_0)$ and $\hat{I}^m(\tau_2, -\eta, \xi, \tau_0)$ of an arbitrary local layer $[\tau_2, \tau_1]$ embedded in the initial uniform slab $[0, \tau_0]$, for downgoing ($\eta > 0$) and upgoing ($\eta < 0$) vision line directions, with the shift parameter $\alpha = 0$.

optical depths $\tau\in[\tau_1,\tau_2]$ within the arbitrary selected local layers with virtual boundaries τ_1 and τ_2 and the simultaneous rotation of the vision lines $\eta \Leftrightarrow -\eta$ (Fig. 1).

Now let's represent these solutions for downgoing ($\eta > 0$) and upgoing ($\eta < 0$) vision line directions as follows:

$$\hat{\hat{I}}^m(\tau',\eta,\xi,\tau_0) = \hat{\hat{I}}^m(\tau_1,\eta,\xi,\tau_0)e^{-\frac{\tau'-\tau_1}{\eta}} + \int\limits_{\tau_1}^{\tau'} \hat{\hat{B}}^m(\tau',\eta,\xi,\tau_0)e^{-\frac{\tau'-\tau''}{\eta}}\frac{d\tau''}{\eta},$$
$$\eta \in [0,1]\cap\xi\in[0,1],\quad [\tau_1,\tau_2]\in[0,\tau_0],\quad \tau_1<\tau_2,\quad m=\overline{0,M}, \tag{1.3}$$

$$\hat{\hat{I}}^m(\tau',-\eta,\xi,\tau_0) = \hat{\hat{I}}^m(\tau_2,-\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau'}{\eta}} + \int\limits_{\tau'}^{\tau_2} \hat{\hat{B}}^m(\tau',-\eta,\xi,\tau_0)e^{-\frac{\tau''-\tau'}{\eta}}\frac{d\tau''}{\eta},$$
$$\eta \in [-1,1]\cap\xi\in[0,1],\quad [\tau_1,\tau_2]\in[0,\tau_0],\quad \tau_1<\tau_2,\quad m=\overline{0,M}, \tag{1.4}$$

where the values $\hat{\hat{B}}^m(\tau',\pm\eta,\xi,\tau_0),(\eta,\xi)>0$ are Fourier harmonics of the initial matrix source functions [2].

From (1.3) and (1.4), at $\tau = \tau_2$ and $\tau = \tau_1$, we obtain the following relations:

$$\hat{\hat{I}}^m(\tau_2,\eta,\xi,\tau_0) = \hat{\hat{I}}^m(\tau_1,\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau_1}{\eta}} + \int\limits_{\tau_1}^{\tau_2} \hat{\hat{B}}^m(\tau',\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau'}{\eta}}\frac{d\tau'}{\eta}, \tag{1.5}$$

$$\hat{\hat{I}}^m(\tau_1,-\eta,\xi,\tau_0) = \hat{\hat{I}}^m(\tau_2,-\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau_1}{\eta}} + \int\limits_{\tau_1}^{\tau_2} \hat{\hat{B}}^m(\tau',-\eta,\xi,\tau_0)e^{-\frac{\tau'-\tau_1}{\eta}}\frac{d\tau'}{\eta}. \tag{1.6}$$

Next we make replacements of the current spatial variables $\tau_2-\tau'=\tau''$ in (1.5) and $\tau'-\tau_1=\tau''$ in (1.6), which allow to pass in the above relations to integrating over the virtual parameter $\alpha=\tau''\geq 0$ that determines the vertical shift of the current optical thickness τ' downward or upward in respect of the boundaries of the arbitrary selected local layer [τ_1, τ_2]:

$$\hat{\hat{I}}^m(\tau_2,\eta,\xi,\tau_0) = \hat{\hat{I}}^m(\tau_1,\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau_1}{\eta}} + \int\limits_{0}^{\tau_2-\tau_1} \hat{\hat{B}}^m(\tau_2-\alpha,\eta,\xi,\tau_0)e^{-\frac{\alpha}{\eta}}\frac{d\alpha}{\eta}, \tag{1.7}$$

$$\hat{\hat{I}}^m(\tau_1,-\eta,\xi,\tau_0) = \hat{\hat{I}}^m(\tau_2,-\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau_1}{\eta}} + \int\limits_{0}^{\tau_2-\tau_1} \hat{\hat{B}}^m(\tau_1+\alpha,-\eta,\xi,\tau_0)e^{-\frac{\alpha}{\eta}}\frac{d\alpha}{\eta}. \tag{1.8}$$

The azimuthal harmonics of matrix source functions $\hat{\hat{B}}^m(\tau_1+\alpha,\eta,\xi,\tau_0)$ and $\hat{\hat{B}}^m(\tau_2-\alpha,-\eta,\xi,\tau_0)$ at arbitrary shifts $\alpha\geq 0$ of the virtual boundaries τ_1 and τ_2 of the current local layers [τ_1, τ_2] are determined as follows:

$$\hat{\hat{B}}^m(\tau_2-\alpha,\eta,\xi,\tau_0) = \frac{\Lambda}{2}\int\limits_{-1}^{1}\hat{p}^m(\eta,\eta')\hat{I}^m(\tau_2-\alpha,\eta',\xi,\tau_0)d\eta' + \frac{\Lambda}{4}\hat{p}^m(\eta,\xi)e^{-\frac{\tau_2-\alpha}{\xi}},$$
$$\eta\in[0,1]\cap\xi\in[0,1],\quad 0\leq\tau_1<\tau_2\leq\tau_0,\quad \alpha\geq 0,\quad m=\overline{0,M}, \tag{1.9}$$

$$\hat{\hat{B}}^m(\tau_1+\alpha,-\eta,\xi,\tau_0)=\frac{\Lambda}{2}\int\limits_{-1}^{1}\hat{D}\hat{p}^m(\eta,\eta')\hat{D}\hat{\hat{I}}^m(\tau_1+\alpha,-\eta',\xi,\tau_0)d\eta'+\frac{\Lambda}{4}\hat{p}^m(-\eta,\xi)e^{-\frac{\tau_1+\alpha}{\xi}},$$
$$\eta\in[0,1]\cap\xi\in[0,1],\quad 0\le\tau_1<\tau_2\le\tau_0,\quad \alpha>0,\quad m=\overline{0,M}. \tag{1.10}$$

The values $\hat{p}^m(\eta,\eta')$ correspond to the azimuthal harmonics of the initial phase matrix functions of scattering $\hat{P}(\cos\gamma)$ and possess the following properties: $\hat{p}^m(-\eta,-\eta')=\hat{D}\hat{p}^m(\eta,\eta')\hat{D}$ and $\hat{p}^m(-\eta,\eta')=\hat{D}\hat{p}^m(\eta,-\eta')\hat{D}$.

Using then (1.5)–(1.6) and (1.9)–(1.10) for the values $\hat{\hat{I}}^m(\tau_2,\alpha,\eta,\xi,\tau_0)$ and $\hat{\hat{I}}^m(\tau_1,\alpha,-\eta,\xi,\tau_0)$ we get the following exact linear Fredholm integral equations of the second kind:

$$\hat{\hat{I}}^m(\tau_2,0,\eta,\xi,\tau_0)=\hat{\hat{I}}^m(\tau_1,0,\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau_1}{\eta}}$$
$$+\frac{\Lambda}{2}\int\limits_{0}^{\tau_2-\tau_1}\left\{\int\limits_{0}^{1}\left[\begin{array}{c}\hat{p}^m(\eta,\eta')\hat{\hat{I}}^m(\tau_2-\alpha,\eta',\xi,\tau_0)\\+\hat{p}^m(\eta,-\eta')\hat{\hat{I}}^m(\tau_2-\alpha,-\eta',\xi,\tau_0)\end{array}\right]d\eta'+\frac{1}{2}\hat{p}^m(\eta,\xi)e^{-\frac{\tau_2-\alpha}{\xi}}\right\}e^{-\frac{\alpha}{\eta}}\frac{d\alpha}{\eta},$$
$$\eta\in[0,1]\cap\xi\in[0,1],\quad 0\le\tau_1<\tau_2\le\tau_0,\quad \alpha\ge0,\quad m=\overline{0,M}, \tag{1.11}$$

$$\hat{\hat{I}}^m(\tau_1,0,-\eta,\xi,\tau_0)=\hat{\hat{I}}^m(\tau_2,0,-\eta,\xi,\tau_0)e^{-\frac{\tau_2-\tau_1}{\eta}}$$
$$+\frac{\Lambda}{2}\int\limits_{0}^{\tau_2-\tau_1}\left\{\int\limits_{0}^{1}\left[\begin{array}{c}\hat{D}\hat{p}^m(\eta,\eta')\hat{D}\hat{\hat{I}}^m(\tau_1+\alpha,-\eta',\xi,\tau_0)\\+\hat{D}\hat{p}^m(\eta,-\eta')\hat{D}\hat{\hat{I}}^m(\tau_1+\alpha,\eta',\xi,\tau_0)\end{array}\right]d\eta'+\frac{1}{2}\hat{p}^m(-\eta,\xi)e^{-\frac{\tau_1+\alpha}{\xi}}\right\}e^{-\frac{\alpha}{\eta}}\frac{d\alpha}{\eta},$$
$$\eta\in[0,1]\cap\xi\in[0,1],\quad 0\le\tau_1<\tau_2\le\tau_0,\quad \alpha\ge0,\quad m=\overline{0,M}. \tag{1.12}$$

Now, introducing instead of (1.1) a more general form of representation of polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)$, taking into account the vertical spatial shift $\alpha\ge0$ at virtual optical levels τ_1 and τ_2 in mirror vision line directions η and –η, we have:

$$\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)=\hat{D}\hat{\hat{I}}^m(\tau_2-\alpha,-\eta,\xi,\tau_0)\pm\hat{\hat{I}}^m(\tau_1+\alpha,\eta,\xi,\tau_0), \tag{1.13}$$

$$\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)=\pm\hat{D}\hat{\hat{I}}^m_{\pm}(\tau_2,\tau_1,-\alpha,-\eta,\xi,\tau_0),$$
$$\eta\in[-1,1]\cap\xi\in[0,1],\quad[\tau_1,\tau_2]\in[0,\tau_0],\quad\alpha\ge0,\quad m=\overline{0,M}. \tag{1.14}$$

In the particular case of virtual levels $\tau_1=0$, $\tau_2=\tau_0$ and shift parameter $\alpha=\tau$ from (1.13)–(1.14) we can define polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau,\eta,\xi,\tau_0)$ and respective invariance relations $\hat{\hat{I}}^m_{\pm}(\tau,\eta,\xi,\tau_0)=\pm\hat{D}\hat{\hat{I}}^m_{\pm}(\tau_0-\tau,-\eta,\xi,\tau_0)$ obtained in [1] at the symmetrical optical levels τ and τ_0–τ for the entire extended layer $\tau\in[0,\tau_0]$, including its external boundaries $\tau=0$ и $\tau=\tau_0$.

Using then (1.13) and (1.14), from (1.11) and (1.12) through some simple transformations we can find exact linear Fredholm integral equations of the second kind for generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ within the arbitrary selected local layers $[\tau_1, \tau_2]$:

$$\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, 0, -\eta, \xi, \tau_0) = \hat{\hat{I}}^m_{\pm}(\tau_2, \tau_1, 0, -\eta, \xi, \tau_0) e^{-\frac{\tau_2 - \tau_1}{\eta}} + \frac{\Lambda}{2} \int_0^{\tau_2 - \tau_1} \left\{ \int_0^1 \left[\begin{array}{c} \hat{p}^m(\eta, \eta') \hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, -\eta', \xi, \tau_0) \\ + \hat{p}^m(\eta, -\eta') \hat{\hat{I}}^m(\tau_1, \tau_2, \alpha, \eta', \xi, \tau_0) \end{array} \right] d\eta' \right. \\ \left. + \frac{1}{2} \left[\hat{D}\hat{p}^m(\eta, \xi) e^{-\frac{\tau_2 - \alpha}{\xi}} \pm \hat{p}^m(-\eta, \xi) e^{-\frac{\tau_1 + \alpha}{\xi}} \right] \right\} e^{-\frac{\alpha}{\eta}} \frac{d\alpha}{\eta},$$

$$\eta \in [-1, 1] \cap \xi \in [0, 1], \quad 0 \le \tau_1 < \tau_2 \le \tau_0, \quad \alpha \in [0, \tau_0], \quad m = \overline{0, M}. \tag{1.15}$$

Note that in the particular case of $\tau_1 = 0$, $\tau_2 = \tau_0$, and $\alpha = \tau_0$ with $\eta < 0$, we have $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0) \equiv 0$, and at $\tau_1 = 0$, $\tau_2 = \tau_0$, $\alpha = 0$ and $\eta > 0$ invariance relations are satisfied for matrix values $\hat{\hat{R}}^m_{\pm}(\eta, \xi, \tau_0) = \hat{\hat{\rho}}^m(\eta, \xi, \tau_0) + \hat{D}\hat{\hat{\sigma}}^m(\eta, \xi, \tau_0)$ of polarized radiation external fields, where $\hat{\hat{\rho}}^m(\eta, \xi, \tau_0)$ and $\hat{D}\hat{\hat{\sigma}}^m(\eta, \xi, \tau_0)$ are the values of matrix brightness coefficient azimuthal harmonics for the initial uniform slab $[0, \tau_0]$.

Therefore, the existence of generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ in a uniform slab of an arbitrary optical thickness $\tau_0 \le \infty$ is determined by the general fundamental property of spatial angular symmetry and invariance of multiple scattered polarized radiation fields at vertical spatial shifts $\alpha \ge 0$ of the virtual boundaries τ_1 and τ_2 of local layers $[\tau_1, \tau_2]$ within the entire given extended layer $[0, \tau_0]$ with simultaneous "mirror" reflections of vision line directions η and $-\eta$.

The virtual parameter $0 \le \alpha \le \tau_0$ of the spatial shift of current optical depths τ describes the vertical motion of the external boundaries $\tau = 0$ and $\tau = \tau_0$ of the given uniform slab into the medium $\tau \in [\tau_1, \tau_2]$ to the boundaries of arbitrary selected local layers $[\tau_1, \tau_2]$ (Fig. 2). Of course, in the above-mentioned particular cases of $\tau_1 = 0$, $\tau_2 = \tau_0$ and $\alpha = \tau_0$ or $\alpha = 0$, the properties of local spatial angular symmetry for the entire extended uniform slab $[0, \tau_0]$ are manifested at the mirror symmetrical $(\tau_0/2)$ optical levels τ and $\tau_0 - \tau$ in the "mirror" vision line directions η and $-\eta$ in accordance with the earlier formulated principle of "mirror" reflection (symmetry) for multiple scattered polarized radiation fields [1, 2].

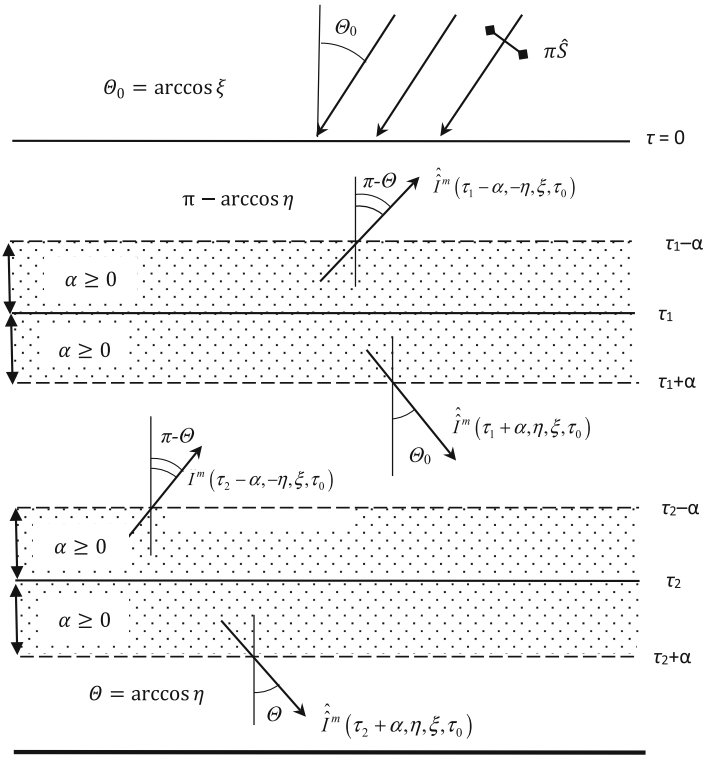


Fig. 2. Azimuthal harmonics of upgoing ($\eta < 0$) and downgoing ($\eta > 0$) polarized radiation in an arbitrary local layer [τ_1, τ_2] with a vertical shift up or down of its upper (τ_1) and lower (τ_2) virtual boundaries by the value of parameter $\alpha \geq 0$.

Taking into account the above analysis, let's now formulate the basic boundary value problem for finding generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$. For that, in the initial matrix equation for polarized radiation transfer, we make changes of variables $\tau' = \tau_1 + \alpha$, $\eta' \to -\eta'$, $\eta \to -\eta$ and $\tau' = \tau_2 - \alpha$, $\eta' \to -\eta'$, $\eta \to -\eta$, which is equipotential to introducing new reference levels of current optical depths τ' relative to the virtual boundaries τ_1 and τ_2 which are respectively measured upward to the upper boundary τ_1 and τ_2 of the given layer $\tau = 0$ for upgoing vision line directions ($\eta < 0$) and downward to its lower boundary $\tau = \tau_0$ for downgoing vision line directions ($\eta > 0$):

$$\eta\frac{\partial\hat{\hat{I}}^m(\tau_1+\alpha,\eta,\xi,\tau_0)}{\partial\alpha}=-\hat{\hat{I}}^m(\tau_1+\alpha,\eta,\xi,\tau_0)$$
$$+\frac{\Lambda}{2}\int\limits_{-1}^{1}\hat{p}^m(\eta,\eta')\hat{\hat{I}}^m(\tau_1+\alpha,\eta',\xi,\tau_0)d\eta'+\frac{\Lambda}{4}\hat{p}^m(\eta,\xi)e^{-\frac{\tau_1+\alpha}{\xi}},$$
$$\eta\in[-1,1]\cap\xi\in[0,1],\quad 0\le\tau_1<\tau_2\le\tau_0,\quad \alpha\in[0,\tau_0],\quad m=\overline{0,M},\tag{1.16}$$

$$\eta\frac{\partial\hat{\hat{I}}^m(\tau_2-\alpha,-\eta,\xi,\tau_0)}{\partial\alpha}=-\hat{\hat{I}}^m(\tau_2-\alpha,-\eta,\xi,\tau_0)$$
$$+\frac{\Lambda}{2}\int\limits_{-1}^{1}\hat{D}\hat{p}^m(\eta,\eta')\hat{D}\hat{\hat{I}}^m(\tau_2-\alpha,-\eta',\xi,\tau_0)d\eta'+\frac{\Lambda}{4}\hat{p}^m(-\eta,\xi)e^{-\frac{\tau_2-\alpha}{\xi}},$$
$$\eta\in[-1,1]\cap\xi\in[0,1],\quad 0\le\tau_1<\tau_2\le\tau_0,\quad \alpha\in[0,\tau_0],\quad m=\overline{0,M}.\tag{1.17}$$

Using then the definition (1.1) of generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)$ and their basic invariance property (1.2), after obvious linear transformations in the integral parts of Eqs. (1.16) and (1.17), based on the changes of angular variables ($\eta\to-\eta'$, $\eta'\to-\eta'$) and the use of angular symmetry properties of Fourier harmonics of phase scattering matrices $\hat{p}^m(\eta,\eta')=\hat{D}\hat{p}^m(-\eta,-\eta')\hat{D}$, we obtain the sought for matrix equation for polarized radiation transfer in an arbitrary local layer [τ_1, τ_2] nested into the initial layer [0, τ_0]:

$$\eta\frac{\partial\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)}{\partial\alpha}=-\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)$$
$$+\frac{\Lambda}{2}\int\limits_{-1}^{1}\hat{p}^m(\eta,\eta')\hat{\hat{I}}^m_{\pm}(\tau_1,\tau_2,\alpha,\eta,\xi,\tau_0)d\eta'$$
$$+\frac{\Lambda}{4}\left[\hat{D}\hat{p}^m(-\eta,\xi)e^{-\frac{\tau_2-\alpha}{\xi}}\pm\hat{p}^m(\eta,\xi)e^{-\frac{\tau_1+\alpha}{\xi}}\right],$$
$$\eta\in[-1,1]\cap\xi\in[0,1],\quad 0\le\tau_1<\tau_2\le\tau_0,\quad \alpha\in[0,\tau_0],\quad m=\overline{0,M}.\tag{1.18}$$

Matrix integral differential Eqs. (1.18) for each current Fourier harmonic "m" generalize the respective polarized radiation transfer equations (obtained in [2] for the entire initial extended layer [0, τ_0]) to the case of virtual local layers [τ_1, τ_2]. Formally, the locally symmetrized transfer Eqs. (1.18) show how the multiple scattered polarized radiation field and its invariance and spatial angular symmetry properties are transformed upon the shift from a given extended uniform slab ($0\le\tau\le\tau_0$) to arbitrary selected local layers $0\le\tau_1<\tau_2\le\tau_0\le\infty$ inside it.

3 Linear Singular Integral Equations for Generalized Polarimetric Invariants

Now let's consider the second approach for finding strict integral relations and linear integral equations that determine azimuthal harmonics of generalized matrix polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$. As noted above, to solve the initial matrix boundary value problem [2], we can apply the classical Case method [5] based on the series representation of sought for values $\hat{\hat{I}}^m(\tau, \eta, \xi, \tau_0)$ by their singular eigenfunctions of continuous and discrete spectra of the initial polarized radiation transfer equation [4].

By applying this method according to consideration in [4] and [6], we obtained linear singular integral equations for vector matrix intensities of polarized radiation at virtual optical levels τ_1 and τ_2 of a uniform semi-infinite slab $\tau_0 = \infty$. Note that in this case, while preserving the general mathematical structure of equations obtained in [4] and [6], vision line angles $\theta = \arccos\eta$ were used as independent variables in [4] (unlike [6]), and the optical depth τ and solar zenith angle $\theta_0 = \arccos\xi$ played the role of numerical modeling parameters. Furthermore, for the full solution of equations [4] it is necessary first of all to solve the auxiliary boundary value problem for polarized radiation diffusely reflected and transmitted by a given uniform slab. Note also that, unfortunately, in [4] quite obvious linear transformations of the received linear singular integral equations were not made, the omission of which did not allow to introduce the notion of generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ into the polarized radiation transfer theory and as a result to investigate, on their basis, local spatial angular properties of the invariance of multiple scattered radiation fields in a uniform slab of an arbitrary optical thickness $\tau_0 \leq \infty$. Later on, this omission was corrected in [6] making use of linear semi-group transformations of Fourier harmonics of the sought for matrix intensities $\hat{\hat{I}}^m(\tau_1 - \alpha, \tau_2 + \alpha, \pm\eta, \xi, \tau_0), (\eta, \xi) > 0$, associated with the spatial shift $\alpha > 0$ of virtual optical depths τ_1 and τ_2 in vertical directions with the simultaneous rotation of vision lines $\eta \Leftrightarrow -\eta$.

By generalizing the analytical results obtained in [4] and [6] to the case of non-symmetrical virtual optical levels τ_1 and τ_2 in a uniform slab of an arbitrary optical thickness $\tau_0 \leq \infty$, we can find the following strict relations of spatial and angular equivalence for multiple scattered downgoing ($\eta > 0$) and upgoing ($\eta < 0$) polarized radiation with the inversion of optical levels $\tau_1 \leftrightarrow \tau_2$ and simultaneous translation of vision line mirror directions $\eta \leftrightarrow -\eta$:

$$
\begin{aligned}
&e^{\frac{\tau_1}{\eta}}\left[\hat{T}^m(\eta)\hat{\hat{I}}^m(\tau_1, \eta, \xi, \tau_0) + \frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m(\tau_1, \eta', \xi, \tau_0)\frac{\eta' d\eta'}{\eta - \eta'}\right] \\
&= e^{\frac{\tau_2}{\eta}}\left[\hat{T}^m(\eta)\hat{\hat{I}}^m(\tau_2, \eta, \xi, \tau_0) + \frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m(\tau_2, \eta', \xi, \tau_0)\frac{\eta' d\eta'}{\eta - \eta'}\right] \\
&+ e^{\frac{\tau_1}{\eta}}\hat{f}^m(\tau_1, \eta, \xi, \tau_0) - e^{\frac{\tau_2}{\eta}}\hat{f}^m(\tau_2, \eta, \xi), \\
&(\eta, \xi) > 0,\ (\tau_1, \tau_2) \in [0, \tau_0],\ m = \overline{0, M},
\end{aligned}
\tag{2.1}
$$

$$e^{\frac{\tau_1}{\eta}}\left[\hat{T}^m(\eta)\hat{I}^m(\tau_2,-\eta,\xi,\tau_0)+\frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{I}^m(\tau_2,-\eta',\xi,\tau_0)\frac{\eta' d\eta'}{\eta-\eta'}\right]$$
$$=e^{\frac{\tau_2}{\eta}}\left[\hat{T}^m(\eta)\hat{I}^m(\tau_1,-\eta,\xi,\tau_0)+\frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{I}^m(\tau_1,-\eta',\xi,\tau_0)\frac{\eta' d\eta'}{\eta-\eta'}\right]+ \quad (2.2)$$
$$+e^{\frac{\tau_2}{\eta}}\hat{f}^m(\tau_1,-\eta,\xi,\tau_0)-e^{\frac{\tau_1}{\eta}}\hat{f}^m(\tau_2,-\eta,\xi,\tau_0),$$
$$(\eta,\xi)>0,\ (\tau_1,\tau_2)\in[0,\tau_0],\ m=\overline{0,M}.$$

The matrix transposition operation in (2.1)–(2.2) is denoted by the symbol "T", the characteristic matrix functions $\hat{\Psi}^m(\eta)$ and $\hat{T}^m(\eta)$ are determined from the following relations:

$$\hat{T}^m(\eta)=\hat{E}(1,1,1,1)+\frac{\Lambda}{2}\eta\int\limits_{-1}^{1}\hat{A}(\eta',\eta')\frac{d\eta'}{\eta'-\eta},\ \hat{\Psi}^m(\eta)=\frac{\Lambda}{2}A^m(\eta,\eta), \quad (2.3)$$

where the parameter Λ is a single scattering albedo, and $\hat{E}(1,1,1,1)$ is a unity matrix. The auxiliary matrix functions $A^m(\eta,\xi)$ are found from the following integral equation:

$$\hat{A}^m(\eta,\xi)=\hat{p}^m(\eta,\xi)+\frac{\Lambda}{2}\eta\int\limits_{-1}^{1}[\hat{p}^m(\eta,\xi)-\hat{p}^m(\eta',\xi)]\frac{\hat{A}^m(\eta,\eta')}{\eta-\eta'}d\eta, \quad (2.4)$$
$$(\eta,\xi)>0,\ m=\overline{0,M}.$$

Matrix functions $\hat{f}^m(\tau,\eta,\xi,\tau_0)$ are determined by single scattering polarized radiation and can be calculated by the following exact formulas:

$$\hat{f}^m(\tau,\eta,\xi,\tau_0)=\begin{cases}\frac{\Lambda}{4}\hat{A}^m(\eta,\xi)\xi\frac{e^{-\frac{\tau}{\xi}}-e^{-\frac{\tau}{\eta}}}{\xi-\eta},\ \eta>0\\ \frac{\Lambda}{4}\hat{A}^m(\eta,\xi)\xi\frac{e^{-\frac{\tau}{\xi}}-e^{-\left(\frac{1}{\xi}-\frac{1}{\eta}\right)(\tau_0-\tau)}e^{-\frac{\tau}{\xi}}}{\xi-\eta},\ \eta<0\end{cases} \quad (2.5)$$
$$\tau=\tau_1\geq 0,\ \tau=\tau_2\leq\tau_0,\ \eta\in[0,\pm1]\cap\xi\in[0,1],\ \overline{0,M}.$$

Note that for obtaining the relation (2.2) we have used the angular symmetry properties of matrix functions $\hat{T}^m(\eta)$ and $\left[\hat{\Psi}^m(\eta)\right]^T$:

$$\hat{T}^m(\eta)=\hat{T}^m(-\eta),\quad \left[\hat{\Psi}^m(-\eta)\right]^T=\left[\hat{\Psi}^m(\eta)\right]^T. \quad (2.6)$$

The above-obtained integral relations (2.1)–(2.2) of the polarized radiation intensities equivalence at the virtual levels τ_1 and τ_2 in the mirror directions η and $-\eta$ can serve as a formal basis for independent – with regard to (1.1) – introduction of

generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$ in a uniform slab of an arbitrary optical thickness $\tau_0 \leq \infty$.

It should, however, be emphasized that in the original work [4] the basic invariance relation (2.1) was considered only in one particular case of $\tau_1 = 0$ and $\tau_2 = \tau$ without considering a possibility of its further use for constructing generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \eta, \xi, \tau_0)$.

This omission makes it impossible to regard the definition of polarimetric invariants (1.13) and invariance relation (1.14) from a general perspective of semi-group linear transformations of the sought for matrix values $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ using the actual spatial and angular variables (τ_1, τ_2, η) at the virtual boundaries $\tau_1 < \tau_2$ of arbitrary selected local layers [τ_1, τ_2] inside an extended medium [0, τ_0] with introducing the parameters of their vertical spatial shift $\alpha \geq 0$. By correcting this omission and introducing in (2.1) and (2.2) the parameter of vertical spatial shift $\alpha \geq 0$, we obtain new invariance relations for $\hat{\hat{I}}^m(\tau_1 + \alpha, \eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_2 - \alpha, -\eta, \xi, \tau_0)$ at virtual boundaries ($\tau_1 + \alpha$) and (τ_2–α) with the simultaneous rotation of the vision line $\eta \to \eta$:

$$
\begin{aligned}
&e^{\frac{\tau_1+\alpha}{\eta}}\left[\hat{T}^m(\eta)\hat{\hat{I}}^m(\tau_1+\alpha,\eta,\xi,\tau_0)+\frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m(\tau_1+\alpha,\eta',\xi,\tau_0)\frac{\eta' d\eta'}{\eta-\eta'}\right]\\
&=e^{\frac{\tau_2-\alpha}{\eta}}\left[\hat{T}^m(\eta)\hat{\hat{I}}^m(\tau_2-\alpha,\eta,\xi,\tau_0)+\frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m(\tau_2-\alpha,\eta',\xi,\tau_0)\frac{\eta' d\eta'}{\eta-\eta'}\right]\\
&+e^{\frac{\tau_1+\alpha}{\eta}}\hat{f}^m(\tau_1+\alpha,\eta,\xi,\tau_0)-e^{\frac{\tau_2-\alpha}{\eta}}\hat{f}^m(\tau_2-\alpha,\eta,\xi,\tau_0),\\
&\eta\in[-1,1]\cap\xi\in[0,1],\quad 0\leq\tau_1<\tau_2\leq\tau_0\leq\infty,\quad \alpha\geq 0,\quad m=\overline{0,M}.
\end{aligned}
\tag{2.7}
$$

Similarly, instead of (2.2) we have the following relation:

$$
\begin{aligned}
&e^{\frac{\tau_1+\alpha}{\eta}}\left[\hat{T}^m(\eta)\hat{\hat{I}}^m(\tau_2-\alpha,-\eta,\xi,\tau_0)+\frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m(\tau_2-\alpha,-\eta',\xi,\tau_0)\frac{\eta' d\eta'}{\eta-\eta'}\right]\\
&=e^{\frac{\tau_2-\alpha}{\eta}}\left[\hat{T}^m(\eta)\hat{\hat{I}}^m(\tau_1+\alpha,-\eta,\xi,\tau_0)+\frac{1}{2}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m(\tau_1+\alpha,-\eta',\xi,\tau_0)\frac{\eta' d\eta'}{\eta-\eta'}\right]\\
&+e^{\frac{\tau_2-\alpha}{\eta}}\hat{f}^m(\tau_1+\alpha,-\eta,\xi,\tau_0)-e^{\frac{\tau_1+\alpha}{\eta}}\hat{f}^m(\tau_2-\alpha,-\eta,\xi,\tau_0),\\
&\eta\in[-1,1]\cap\xi\in[0,1],\quad 0\leq\tau_1<\tau_2\leq\tau_0\leq\infty,\quad \alpha\geq 0,\quad m=\overline{0,M}.
\end{aligned}
\tag{2.8}
$$

Thus, Eqs. (2.7) and (2.8) form a system of two matrix linear integral singular equations with regard to the sought for values $\hat{\hat{I}}^m(\tau_1 + \alpha, \eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_2 - \alpha, -\eta, \xi, \tau_0)$ at two virtual non-symmetrical optical levels $(\tau_1 + \alpha)$ and $(\tau_2 - \alpha)$ in the mirror vision line directions η и –η.

Let's now introduce into consideration the generalized polarimetric invariants $\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ according the relations (1.13). By making respective transformations in (2.7) and (2.8), instead of the above-mentioned system of equations we obtain separate linear singular integral equations for finding Fourier harmonics of the generalized polarimetric invariants $\hat{\hat{I}}^m_{+}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ и $\hat{\hat{I}}^m_{-}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$:

$$\begin{aligned}
&\hat{T}^m(\eta)\left[e^{\frac{\tau_1+\alpha}{\eta}}\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0) \mp e^{\frac{\tau_2-\alpha}{\eta}}\hat{D}\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, -\eta, \xi, \tau_0)\right] \\
&= \frac{1}{2}e^{\frac{\tau_1+\alpha}{\eta}}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)\frac{\eta' d\eta'}{\eta' - \eta} \\
&\mp \frac{1}{2}e^{\frac{\tau_2-\alpha}{\eta}}\int\limits_{-1}^{1}\left[\hat{\Psi}^m(\eta')\right]^T\hat{D}\hat{\hat{I}}^m_{\pm}(\tau_1, \tau_2, \alpha, -\eta, \xi, \tau_0)\frac{\eta' d\eta'}{\eta' - \eta} \\
&\pm e^{\frac{\tau_1+\alpha}{\eta}}\hat{f}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0) \mp e^{\frac{\tau_2-\alpha}{\eta}}\hat{D}\hat{f}^m_{\pm}(\tau_1, \tau_2, \alpha, -\eta, \xi, \tau_0), \\
&\eta \in [-1, 1] \cap \xi \in [0, 1], \quad 0 \le \tau_1 < \tau_2 \le \tau_0 \le \infty, \quad \alpha \ge 0, \quad m = \overline{0, M},
\end{aligned} \tag{2.9}$$

where the values $\hat{f}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ are defined as follows:

$$\hat{f}^m_{\pm}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0) = \hat{D}\hat{f}^m(\tau_2 - \alpha, -\eta, \xi, \tau_0) \mp \hat{f}^m(\tau_1 + \alpha, \eta, \xi, \tau_0). \tag{2.10}$$

Note that the invariance property (1.14) allows to consider regular solutions $\hat{\hat{I}}^m_{+}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m_{-}(\tau_1, \tau_2, \alpha, \eta, \xi, \tau_0)$ in more narrow domains of actual variables (τ_1, τ_2, η) as compared to the system of Eqs. (2.7)–(2.8), namely: $\tau \in (\tau_2 - \tau_1)$, $\eta \in [0, 1]$, $\alpha \ge 0$ or alternatively $\tau \in \frac{1}{2}(\tau_2 - \tau_1)$, $\eta \in [-1, 1]$, $\alpha \ge 0$. Furthermore, in case of numerical solutions of certain matrix linear singular integral Eqs. (2.9), e.g. by the method of spatial angular discretization or by using the Gauss-Seidel relaxation iterative method, the rank of respective linear algebraic systems is 2 times lower than the rank of similar systems of algebraic Eqs. (2.7)–(2.8). However, as in the scalar case [1], matrix linear singular integral Eqs. (2.9) can have non-unique solutions, and to eliminate them it is necessary to apply precise additional integral relations while using characteristic roots $\xi = 1/\hat{k}_m$ of matrix equations $\hat{T}^m(\xi) = 0$, $\xi \in [0, 1]$, $m = \overline{0, M}$.

Now let's consider some formal mathematical peculiarities of Eq. (2.9). Naturally, any regular solution of the initial boundary value problem in the polarized radiation transfer theory, with respective boundary conditions, must satisfy the said matrix linear singular integral equation at arbitrary optical depths τ_1 and τ_2 in the mirror vision line directions η and $-\eta$. Yet this equation cannot directly provide a solution to the initial boundary value problem, since it was derived without using required boundary conditions. Therefore, Eq. (2.9), as well as Eqs. (2.7) and (2.8) can be viewed only as some new calibration integral relations to which any regular solutions of the initial equation for polarized radiation transfer having physical content must satisfy. On the other hand, taking into account relevant boundary conditions, they can serve as an alternative to the initial boundary value problem in the theory of polarized radiation

transfer in a uniform slab of an arbitrary optical thickness $\tau_0 < \infty$. The realization of such a capability, however, is a complex mathematical task as it requires additional and cumbersome investigation of the problem of existence and uniqueness of sought for regular solutions.

If we consider one of values $\hat{\hat{I}}^m(\tau_1 + \alpha, \eta, \xi, \tau_0)$ or $\hat{\hat{I}}^m(\tau_2 - \alpha, -\eta, \xi, \tau_0)$ to be known, then relations (2.7) and (2.8) will turn into exact matrix linear singular equations for direct finding of the other unknown value. Finally, it should be emphasized that the obvious generalization of the scalar approach [8] to the case of polarization of multiple scattered radiation allows to use the known values $\hat{\hat{I}}^m(\tau_1, \eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_2, -\eta, \xi, \tau_0)$ at fixed optical depths τ_1 and τ_2 for finding respective azimuthal harmonics of matrix intensities $\hat{\hat{I}}^m(\tau, \pm\eta, \xi, \tau_0)$ at summarized optical depths $\tau = \tau_1 + \tau_2$. The scalar variant [7], however, did not point to an algorithm of finding the values $\hat{\hat{I}}^m(\tau_1, \eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_2, -\eta, \xi, \tau_0)$ by the known Fourier harmonics $\hat{\hat{I}}^m(\tau_1 + \tau_2, \pm\eta, \xi, \tau_0)$, $\eta > 0$. In this respect, linear singular integral Eqs. (2.7) and (2.8) eliminate that omission and enable us to close the scheme for calculating values $\hat{\hat{I}}^m(\tau_1, \pm\eta, \xi, \tau_0)$, $\hat{\hat{I}}^m(\tau_2, \pm\eta, \xi, \tau_0)$ and $\hat{\hat{I}}^m(\tau_1 + \tau_2, \pm\eta, \xi, \tau_0)$, $(\eta, \xi) > 0$, $m = \overline{0, M}$ into a single self-consistent computational cycle.

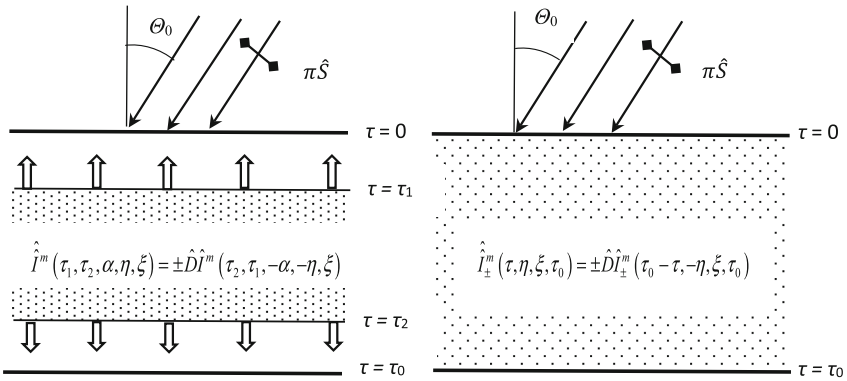


Fig. 3. Preserving the property of spatial angular symmetry and invariance of polarized radiation fields in an arbitrary local layer $[\tau_1, \tau_2]$ when it is extended to the boundaries $\tau_1 = 0$ и $\tau_2 = \tau_0$ of the initial uniform slab of a finite optical thickness $\tau_0 < \infty$.

4 Conclusion

In conclusion, note that the obtained new linear singular integral Eq. (2.9) expresses the general fundamental property of spatial angular invariance of internal polarized radiation fields in a uniform slab of an arbitrary optical thickness ($\tau_0 \leq \infty$) relative to a semigroup of linear spatial angular transformations of regular solutions to the initial

boundary value problem in the polarized radiation transfer theory, which are associated with a spatial vertical shift of the current optical levels $\tau_i \leftrightarrow \tau_{i+1}$ and angular rotation of respective vision lines $\eta \leftrightarrow -\eta$. In that case, if the basic invariance relation (1.14) is true for a current local layer $[\tau_i, \tau_{i+1}]$ inside the given uniform slab of an arbitrary optical thickness ($\tau_0 \leq \infty$), then it must preserve for the entire extended uniform layer within the limits $\tau \in [0, \infty]$, and vice versa (Fig. 3).

Of course, inside a semi-infinite uniform medium ($\tau_0 = \infty$) this fundamental property of spatial angular symmetry and invariance formally manifests itself only in the case of above-mentioned linear transformations of polarized radiation intensity [6]. In media of a finite optical thickness ($\tau_0 < \infty$), having fixed top and bottom boundaries, this property manifests itself as an obvious consequence of the mirror reflection of polarized radiation fields at symmetrical (with respect to the layer middle $\tau_0/2$) levels (τ) and (τ_0–τ) in the mirror vision line directions (η) and ($-\eta$). It should emphasized that the above conception can also be applied for generalized photometrical invariants of the uniform atmospheric slab – arbitrary underlying surface system by using the broad interpretation of the Ambarzumian – Chandrasekhar invariance principle [9].

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