Kalman Filter Employment in Image Processing

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Abstract. The Kalman filter is a classical algorithm of estimation and control theory. Its use in image processing is not very well known as it is not its typical application area. The paper deals with the presentation and demonstration of selected possibilities of using the Kalman filter in image processing. Particular attention is paid to problems of image noise filtering and blurred image restoration. The contribution presents the reduced update Kalman filter algorithm, that can be used to solve both the tasks. The construction of the image model, which is the necessary first step prior to the application of the algorithm itself, is briefly mentioned too. The described procedures are then implemented in the MATLAB software and the results are presented and discussed in the paper.

Keywords: Kalman filter · Image processing · Image noise filtering · Blurred image restoration

1 Introduction

The Kalman filter represents a theoretical basis for various recursive methods in the examination of stochastic (linear) dynamic systems. The algorithm is based on the idea that the unknown state of the system can be estimated using certain measured data. The filter is named after Rudolf Emil Kalman, a Hungarian mathematician living in the United States, who published it in the article [15] in 1960. In the next period, various algorithms based on the nature of the Kalman filter have been derived by various authors. These algorithms are all referred to as Kalman filters and can be suitably used in certain specific situations, for example, resulting from failure to meet some theoretical assumptions of the classical Kalman filter in solving practical problems.

The Kalman filter is used in various applications. Location of moving objects and navigation belong to its most important application domains – the Kalman filter or generally Kalman filters are used for example in global satellite positioning systems (GPS etc.), in radars, in robot control and navigation, in autopilots or autonomous vehicles. They are also used in the area of computer vision for tracking the movement of objects in a sequence of video frames, in augmented and virtual reality, etc. The space project Apollo which dates back to

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1960s included one of the first applications of the Kalman filter in the area of navigation and control. Other application domains include time series analysis, econometrics, signal processing, weather forecasting and many others.

This paper presents some possibilities of using the Kalman filter in image processing. It is organized as follows. Section 2 provides a description of the Kalman filter algorithm and the principles of its functioning. Section 3 deals with its application to image noise filtering and blurred image restoration. Section 4 demonstrates the experimental results achieved. Section 5 is the final summary.

2 Kalman Filter

The Kalman filter is a tool for estimating the state of a stochastic linear dynamic system using measured data corrupted by noise. The estimate produced by the Kalman filter is statistically optimal in some sense (for example it minimizes the mean square error, see [25] for more details). The principle of the application of the filter is shown in the following Fig. 1.

The Kalman filter works with all available information, i.e. it uses all available measured data, system model together with statistical description of its inaccuracies, noise and measurement errors as well as information about initial conditions and initial state of the system.

2.1 Algorithm of the Kalman Filter

Let us consider a stochastic linear dynamic system in discrete time represented by the following model (it is assumed here that the system has no inputs)

$$\boldsymbol{x}_k = \boldsymbol{\Phi}_{k-1}\boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1}, \tag{1}$$

$$\boldsymbol{z}_k = \mathbf{H}_k\boldsymbol{x}_k + \boldsymbol{v}_k. \tag{2}$$

Equation (1), referred to as state equation, describes the dynamics of the system, the vector $\boldsymbol{x}_k \in \mathbb{R}^n$ is an (unknown) vector of the system state at time t_k, the matrix $\boldsymbol{\Phi}_{k-1} \in \mathbb{R}^{n\times n}$ represents the dynamic evolution of the system between time t_{k-1} and t_k. Equation (2) is called measurement equation, the vector $\boldsymbol{z}_k \in \mathbb{R}^m$ is called the system output vector, measurement vector or observation vector; the matrix $\mathbf{H}_k \in \mathbb{R}^{m\times n}$ describes the relationship between the state of the system and measurements. The vectors $\boldsymbol{x}_k$ and $\boldsymbol{z}_k$, $k = 0, 1, 2, \ldots$, may be treated as random variables and their sequences $\{\boldsymbol{x}_k\}$ and $\{\boldsymbol{z}_k\}$ considered as random processes as we deal with a stochastic system.

$\{\boldsymbol{w}_k\}$ and $\{\boldsymbol{v}_k\}$ are random noise processes; these processes are assumed to be uncorrelated Gaussian processes with zero mean and covariance matrices $\mathbf{Q}_k$ resp. $\mathbf{R}_k$ at time t_k (the processes have Gaussian white noise properties).

Further, let $\boldsymbol{x}_0$ be a random variable with a Gaussian (normal) distribution with known mean $\boldsymbol{x}_0$ and known covariance matrix $\mathbf{P}_0$. Moreover, let $\boldsymbol{x}_0$ and

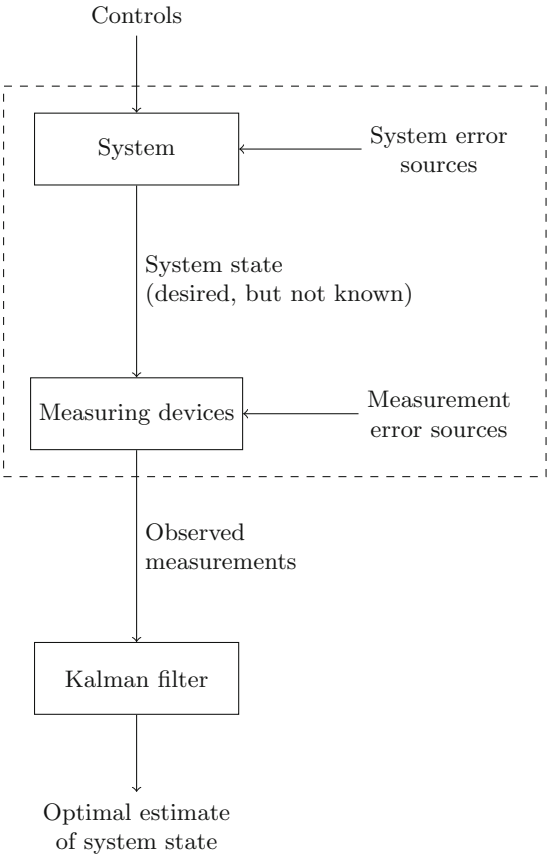


Fig. 1. Scheme of applying the Kalman filter

both the noises be mutually uncorrelated. These can be expressed for all t_k as follows

$$\begin{aligned}
\mathrm{E}\langle \boldsymbol{w}_k \rangle &= \boldsymbol{0}, \\
\mathrm{E}\langle \boldsymbol{v}_k \rangle &= \boldsymbol{0}, \\
\mathrm{E}\langle \boldsymbol{w}_{k_1} \boldsymbol{w}_{k_2}^T \rangle &= \mathbf{Q}_{k_1} \delta(k_2 - k_1), \\
\mathrm{E}\langle \boldsymbol{v}_{k_1} \boldsymbol{v}_{k_2}^T \rangle &= \mathbf{R}_{k_1} \delta(k_2 - k_1), \\
\mathrm{E}\langle \boldsymbol{w}_{k_1} \boldsymbol{v}_{k_2}^T \rangle &= \mathbf{0}, \\
\mathrm{E}\langle \boldsymbol{x}_0 \boldsymbol{w}_k^T \rangle &= \mathbf{0}, \\
\mathrm{E}\langle \boldsymbol{x}_0 \boldsymbol{v}_k^T \rangle &= \mathbf{0},
\end{aligned}$$

where the symbol δ denotes the Kronecker delta

$$\delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

The goal of the Kalman filter is to find an estimate of the state vector $\boldsymbol{x}_k$ at time t_k, denoted as $\hat{\boldsymbol{x}}_k$, so that this estimate is optimal (for example in the sense of minimizing the mean square error).

The Kalman filter algorithm is recursive; the calculation at general time t_k consists of two main steps. The first step is the calculation of the a priori estimate $\hat{\boldsymbol{x}}_{k(-)}$ at time t_k by substituting the a posteriori estimate from time t_{k-1} into the deterministic part of the state equation of the model. In the second step this estimate is adjusted using the measurement taken at time t_k, which results in obtaining the a posteriori estimate $\hat{\boldsymbol{x}}_{k(+)}$ at time t_k.

The following equation can be written for the a priori estimate of the state vector $\hat{\boldsymbol{x}}_{k(-)}$ at time t_k. The uncertainty of this estimate is expressed by the a priori error covariance matrix $\mathbf{P}_{k(-)}$

$$\begin{aligned} \hat{\boldsymbol{x}}_{k(-)} &= \boldsymbol{\Phi}_{k-1}\hat{\boldsymbol{x}}_{k-1(+)}, \\ \mathbf{P}_{k(-)} &= \boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1(+)}\boldsymbol{\Phi}_{k-1}^T + \mathbf{Q}_{k-1}. \end{aligned}$$

Then, after obtaining the measurement $\boldsymbol{z}_k$ the a posteriori estimate of the state vector $\hat{\boldsymbol{x}}_{k(+)}$ is calculated as a combination of the a priori estimate and the difference between the actual and expected value of the measurement weighted by the matrix $\mathbf{K}_k$ (called the Kalman gain). Its uncertainty is expressed by the a posteriori error covariance matrix $\mathbf{P}_{k(+)}$

$$\begin{aligned} \hat{\boldsymbol{x}}_{k(+)} &= \hat{\boldsymbol{x}}_{k(-)} + \mathbf{K}_k[\boldsymbol{z}_k - \mathbf{H}_k\hat{\boldsymbol{x}}_{k(-)}], \\ \mathbf{P}_{k(+)} &= \mathbf{P}_{k(-)} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k(-)}, \\ \mathbf{K}_k &= \mathbf{P}_{k(-)}\mathbf{H}_k^T[\mathbf{H}_k\mathbf{P}_{k(-)}\mathbf{H}_k^T + \mathbf{R}_k]^{-1}. \end{aligned}$$

The detailed derivation of the mentioned equations of the Kalman filter can be found, for example, in [11] – this derivation uses the orthogonality principle, which Kalman used in his original article [15]. Over time, other approaches to derivation based on innovations or Bayesian statistics have been used by other authors, they can be found, for example, in [3,25].

The Kalman filter can also be generalized for systems with inputs (see e.g. [7, p. 28]) and there also exists a variant for continuous-time systems – the Kalman-Bucy filter [16].

A more detailed discussion of the algorithm, its properties and theoretical assumptions is offered, for example, by [11,25,28]; practical aspects of the filter implementation are discussed, for example, in [28].

3 Kalman Filter and Image Processing

A grayscale digital image can be naturally represented by a two-dimensional matrix; its elements then express the intensity values of individual pixels. Using

the Kalman filter in image processing tasks thus requires to extend the concept of the Kalman filter from random processes to two-dimensional random fields.

3.1 Reduced Update Kalman Filter

One of the Kalman filter modifications intended and designed for this purpose is the reduced update Kalman filter (RUKF) published by Woods and Radewan [32]. This algorithm was originally designed to filter noise in images, but later Woods and Ingle [33] further developed it and extended it for blurred image restoration.

The algorithm is based on the following two-dimensional autoregressive (2D AR) image model (state equation)

$$x(i,j) = \sum_{(k,l)\in\mathcal{D}} \phi(k,l)x(i-k,j-l) + w(i,j),$$

where the notation $x(i,j)$ represents a pixel of the ideal image located at the position (i,j), $w(i,j)$ denotes system "noise" corresponding to model inaccuracies and $\phi(k,l)$ represents the corresponding parameter of the autoregressive model, it is assumed that $\mathcal{D} = \{k \geq 0, l \geq 0\} \cup \{k > 0, l < 0\}$. Section 3.2 provides more detailed information about the image model identification.

Corruption of the image by additive measurement noise can be modelled by the following scalar equation (measurement equation)

$$z(i,j) = x(i,j) + v(i,j),$$

where $z(i,j)$ denotes a pixel of the noisy image and $v(i,j)$ is noise incurred usually due to the technical principles of image obtaining.

Both w and v possess properties of Gaussian white noise according to the assumptions required by the Kalman filter.

Sometimes, the image may be degraded in addition to noise also by blurring. The measurement equation has in this case the following form

$$z(i,j) = \sum_{(k,l)} h(k,l)x(i-k,j-l) + v(i,j),$$

where h represents blur of the image caused by, for example, motion or poorly focused camera optics. The above equation can also be seen as the expression of the two-dimensional discrete convolution of the image x with the convolution kernel h. More details about image blur modeling can be found, e.g., in [31].

The already mentioned reduced update Kalman filter can be used to solve both these tasks. This algorithm is based on sequential scanning of the image (pixel by pixel) starting at the upper-left corner of the image and continuing on a line by line basis. At any given moment, the currently processed pixel can be perceived as the "presence", the pixels already processed as the "past" and the upcoming pixels waiting for processing as the "future", see Fig. 2. This approach transforms a two-dimensional problem into a one-dimensional problem

and consequently the classical Kalman filter can be used. The problem, however, is that the state vector, which is made up of individual pixels of an image, takes on a large number of elements, resulting in high computational costs. The RUKF solves this situation by not working always with the whole image but applying the Kalman filter only to a certain area surrounding the currently processed pixel. A detailed description of the algorithm and derivation of the filter equations can be found in [31–33]. Computational demands can also be reduced by taking advantage of the fact that the Kalman gain usually achieves a steady-state value after several iterations of the algorithm, and consequently, it is no longer necessary to calculate the Kalman gain along with the error covariance equations in each iteration, as it can be approximated by this steady-state value. This modification of the algorithm is called the steady-state RUKF [31].

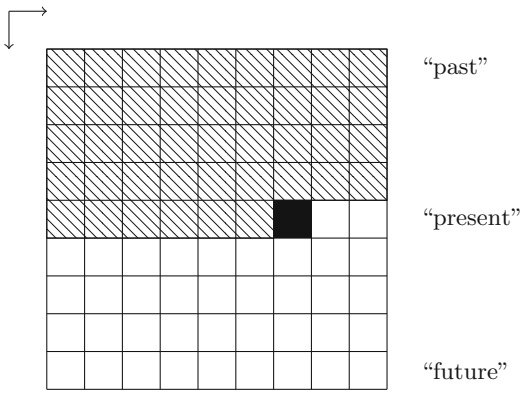


Fig. 2. Image scanning

3.2 Construction of the Image Model

Woods and Radewan [32] assumed in their original paper that the image model is known. In practice, this is not usually the case.

Image modeling is based on finding the order and parameters of a two-dimensional autoregressive model and estimating the variance of the corresponding noise (model inaccuracies).

The first step in constructing the model lies in determination of its order, usually denoted $\theta(p, q)$. Bouzouba and Radouane [5] suggest using the maximum entropy principle to solve this problem. Let

$$\hat{e}_\theta = z - \hat{x}_\theta$$

be the estimation error defined as the difference between the noisy degraded image and the estimate of the ideal image obtained by the RUKF algorithm using

the model of order $\theta(p,q)$. For the pixels of the image, the following definition of the probability density function of the error can be introduced

$$\hat{g}_\theta(i,j) = \frac{\hat{e}_\theta(i,j)}{\sum_i \sum_j \hat{e}_\theta(i,j)},$$

$\sum_i \sum_j \hat{g}_\theta(i,j) = 1$ and $0 < \hat{g}_\theta(i,j) \leq 1$. Next, let

$$G = \{\hat{g}_\theta : \theta = (1,1); (1,2); (2,1); ...; (p_{max}, q_{max})\}$$

be the set containing possible estimates of $\hat{g}_\theta$ for various model orders θ. According to the principle of maximum entropy [13,14] the optimal estimate of the probability density function is the estimate $\hat{g}^*_\theta$, having the maximum entropy among all estimates $\hat{g}_\theta$. The entropy of $\hat{g}_\theta$ can be expressed by the formula

$$H(\hat{g}_\theta) = -\sum_i \sum_j \hat{g}_\theta(i,j) \log(\hat{g}_\theta(i,j)),$$

then, the following holds for the optimal estimate of the probability density function

$$H(\hat{g}^*_\theta) = \max\{H(\hat{g}_\theta) : \hat{g}_\theta \in G\}.$$

The choice of the appropriate order of the model therefore consists in estimating the probability density function $\hat{g}_\theta$ for various possible orders and then selecting the order θ, for which $\hat{g}_\theta$ corresponds to the maximum entropy.

Next, it is necessary to calculate estimates of model parameters and noise variance. Different approaches or algorithms can be used, such as variants of least squares method, correlation-based methods, or approaches enabling to estimate model parameters simultaneously with image filtering. A description and comparison of these approaches is given in [18].

Various forms of models (based on areas covered by the models) can be met. Nonsymmetric half-plane (NSHP) or quarter plane (QP) form are usually used.

4 Practical Demonstration and Obtained Results

For a practical demonstration of the RUKF algorithm, a photograph of the Large Square in the city of Hradec Kralove (the Czech Republic) was used, see Fig. 3a.

Additive Gaussian white noise was added to the photograph in the first experiment, as is shown in Fig. 3b. The amount of noise present in an image can be expressed using the signal-to-noise ratio (SNR), in the case of Fig. 3b the SNR is 10.77 dB.

Prior to the RUKF application, the model of the image was constructed. First, the order of the model was determined using the principle of maximum entropy. The highest value of entropy was associated with the model of order $(2,2)$. The model coefficients (together with the estimation of variance of the corresponding inaccuracies) were calculated using the method described in [5].

Variance of measurement noise was considered known, according to [5], this is not an unrealistic assumption in practice.

Subsequently, the RUKF was applied using the proposed model. The result is shown in Fig. 3c, the SNR rose to 15.36 dB, so the improvement in SNR (ISNR) is 4.59 dB.

In the second experiment, the original photograph was degraded also by blurring caused by motion in the horizontal direction as Fig. 4b shows. The result obtained by applying the RUKF is shown in Fig. 4c.

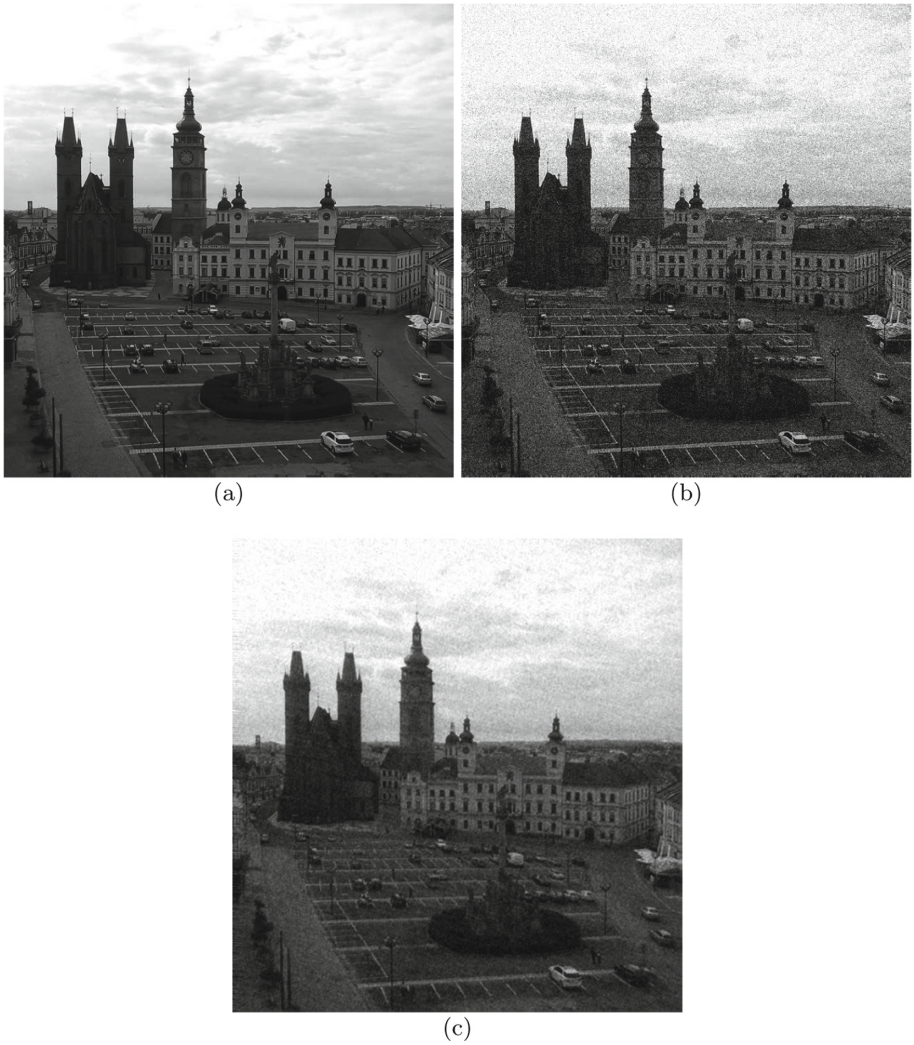


Fig. 3. Image noise filtering by the RUKF: (a) original ideal image, (b) image degraded by additive noise, (c) result obtained using the RUKF



(a) (b)



(c)

Fig. 4. Blurred image restoration by the RUKF: (a) original ideal image, (b) image degraded by blur, (c) result obtained using the RUKF

As the figures show, after the application of the RUKF algorithm, visual improvement is evident. In the first experiment, the effect of noise is suppressed in the resulting image, but also the sharpness of its details is reduced. By choosing suitable filter parameters (noise variances), it would be possible to find the desired compromise between noise filtering and maintaining sharpness. In the case of the second experiment, blurring was removed from the image, but as a result, artifacts were created, that are most noticeable near sharp transitions (edges) in the image.

The Kalman filter resp. the RUKF algorithm provides optimal results from a theoretical point of view (for example in the sense of minimizing the mean square error), but in practice, the problem may be ignorance of the actual image model or the blurring process model, failure to meet noise assumptions (for example Gaussian white noise assumption), etc., moreover, the complete RUKF algorithm has high computational demands. As a consequence of the non-fulfillment of theoretical assumptions or the use of a certain modification of the algorithm having, for example, more favorable computational properties, the results obtained in practice are no longer optimal.

The reduced update Kalman filter algorithm is a basic approach to applying the Kalman filter to images. Over time, other authors have derived various modifications of the RUKF algorithm and slightly different approaches employing the Kalman filter, which are more suitable, for example, in certain real situations that do not meet the basic theoretical assumptions or in terms of computational performance. In addition to the classical Kalman filter, algorithms can also be based on other Kalman filters, such as the extended Kalman filter, the robust Kalman filter, the adaptive Kalman filter, etc. Examples are [32] (Kalman strip filter), [36] (fast modified RUKF), [19] (reduced order model Kalman filter), [1,2,6,8,12,22,34,35].

In a general comparison with other methods for image restoration, the mentioned algorithms based on the Kalman filter can be included among the classical methods. These methods are based on mathematical formulas and algorithms known for many years. The RUKF achieves similar results compared to these methods. Recently, new approaches have emerged, that are now widely used, especially those based on neural networks. In general, these approaches usually achieve better results compared to mathematical methods, but an accurate description of their functioning is hard to follow and also training can be difficult (they require a large amount of training data, computational power and knowledge). Currently, algorithms combining classical and new approaches are achieving interesting results - and even here the Kalman filter finds its application [23].

In practice, algorithms based on the Kalman filter are used, for example, in the restoration of satellite and radar images [17,24,30] or various medical images (MRI, CT, ultrasound, ...) [4,9,20,27,29] and can be used even for color image restoration [10,21,26]. Moreover, the Kalman filter can be employed also in related tasks such as estimating image model parameters [37] etc.

5 Conclusion

The Kalman filter can find its employment also in the filtering of two-dimensional signals and data. The paper presented possibilities of its use in the field of image processing, namely in solving the problems of image noise filtering and blurred image restoration. The construction of the image model using the maximum entropy principle for model order determination was also mentioned. The use of the described procedures was illustrated by the example and the obtained results were discussed.

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