

Multivalue Almost Collocation Methods with Diagonal Coefficient Matrix

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Abstract. We introduce a family of multivalue almost collocation methods with diagonal coefficient matrix for the numerical solution of ordinary differential equations. The choice of this type of coefficient matrix permits a reduction of the computational cost and a parallel implementation. Collocation gives a continuous extension of the solution which is useful for a variable step size implementation. We provide examples of A-stable methods with two and three stages and order 3.

Keywords: Multivalue methods · Almost collocation · Schur analysis

1 Introduction

Consider the initial value problem:

$$\begin{cases} y'(t) = f(y(t)), \, t \in [t_0, T], \\ y(t_0) = y_0, \end{cases} \tag{1}$$

$f : \mathbb{R}^k \to \mathbb{R}^k$. Multivalue methods are a large class of numerical methods used to solve (1). Classical methods for the solution of ordinary differential equations, such as Runge Kutta and linear multistep methods, are special cases of these methods [3,5,6,28,45]. Multivalue methods have also been treated as geometric numerical integrators in [4,25,28].

Multivalue methods are characterized by the abscissa vector $\mathbf{c} = [c_1, c_2, ..., c_m]^T$ and four coefficient matrices $\mathbf{A} = [a_{ij}]$, $\mathbf{U} = [u_{ij}]$, $\mathbf{B} = [b_{ij}]$ and $\mathbf{V} = [v_{ij}]$, where:

$$\mathbf{A} \in \mathbb{R}^{m \times m}, \qquad \mathbf{U} \in \mathbb{R}^{m \times r}, \qquad \mathbf{B} \in \mathbb{R}^{r \times m}, \qquad \mathbf{V} \in \mathbb{R}^{r \times r}.$$

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On the uniform grid $t_n = t_0 + nh, n = 0, 1, ..., N, Nh = T - t_0$, the method takes the form:

$$\begin{aligned} Y_i^{[n]} &= h \sum_{j=1}^{m} a_{ij} f\left(Y_j^{[n]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \; i = 1, 2, ..., m, \\ y_i^{[n]} &= h \sum_{j=1}^{m} b_{ij} f\left(Y_j^{[n]}\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \quad i = 1, 2, ..., r, \end{aligned} \tag{2}$$

$n = 0, ..., N$, where m is the number of internal stages and r is the number of external stages.

Multivalue methods are defined by a starting procedure $\mathcal{S}_h$ for the computation of the starting vector, a forward procedure $\mathcal{G}_h$, which updates the vector of the approximations at each step point and a finishing procedure $\mathcal{F}_h$, which permits to compute the corresponding numerical solution. These methods can be extended using collocation in order to obtain a smooth solution. Collocation is a technique which approximates the solution with continuous approximants belonging to a finite dimensional space (usually algebraic polynomials). The approximation satisfies interpolation conditions at the grid points and satisfies the differential equations on the collocation points [46,48,49,51,52,56]. Those methods are very effective because they permit to avoid the order reduction typical of Runge Kutta methods, also in presence of stiffness. Stiff problems arise in many relevant mathematical models [44,50,55], therefore they are object of wide attention in the literature, see [7,43,47,54] and references therein.

Because of the implicitness of such methods, the computational cost of the integration process is strictly connected to the numerical solution of non linear systems of external stages at each time step of dimension mk, where k is the dimension of system (1) and m is the number of stages. We focus on the development of methods with diagonal coefficient matrix $\mathbf{A}$ in (2), for which the nonlinear system of mk equations reduces to m independent systems of dimension k, thus it is possible to reduce the computational effort and to parallelize the method.

In order to build A-stable methods, it is not possible to impose all the collocation conditions and thus we consider almost collocation [27,34]. Collocation and almost collocation methods are widely used also for the solution of integral and integro-differential equations [9,17,23].

The organization of this paper is as follows. In Sect. 2 we summarize multivalue methods, describing their formulation and some results about order conditions. In Sect. 3 we discuss the construction of almost collocation methods with diagonal coefficient matrix. In Sect. 4 we present some examples of methods with two and three stages. In Sect. 5 numerical results are provided. Finally, in Sect. 6 some concluding remarks are given and plans for future research are outlined.

2 Multivalue Collocation Methods

Multivalue collocation methods described in [39] are of the form (2) where the external stages have the Nordsieck form:

$$y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} \approx \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ \vdots \\ h^{r-1}y^{r-1}(x_n) \end{bmatrix} \tag{3}$$

and the piecewise collocation polynomial:

$$P_n(t_n + \theta h) = \sum_{i=1}^{r} \alpha_i(\theta) y_i^{[n]} + h \sum_{i=1}^{m} \beta_i(\theta) f(P_n(t_n + c_i h)), \quad \theta \in [0,1], \tag{4}$$

provides a dense approximation to the solution of (1). We impose the following interpolation conditions:

$$P_n(t_n) = y_1^{[n]}, \quad hP_n'(t_n) = y_2^{[n]}, \quad \ldots \quad h^{r-1}P_n^{(r-1)}(t_n) = y_r^{[n]},$$

and collocation conditions

$$P_n'(t_n + c_i h) = f(P_n(t_n + c_i h)), \quad i = 1, 2, .., m.$$

We can observe that the polynomial (4) has globally class C^{r-1} while most interpolants based on Runge-Kutta methods only have global C^1 continuity [41,42].

So, the matrices of multivalue methods assume the following form:

$$\mathbf{A} = [\beta_j(c_i)]_{i,j=1,\ldots,m}, \qquad \mathbf{U} = [\alpha_j(c_i)]_{i=1,\ldots,m,j=1,\ldots,r},$$

$$\mathbf{B} = \left[\beta_j^{(i-1)}(1)\right]_{i=1,\ldots,m,j=1,\ldots,r}, \ \mathbf{V} = \left[\alpha_j^{(i-1)}(1)\right]_{i,j=1,\ldots,r}.$$

We, now, summarize some important results regarding the order of the method [39].

Theorem 1. *A multivalue collocation method given by the approximation* $P_n(t_n + \theta h)$ *in* (4), $\theta \in [0,1]$, *is an approximation of uniform order* p *to the solution of the well-posed problem* (1) *if and only if*

$$\alpha_1(\theta) = 1 \tag{5}$$

$$\frac{\theta^\nu}{\nu!} - \alpha_{\nu+1}(\theta) - \sum_{i=1}^{m} \frac{c_i^{\nu-1}}{(\nu-1)!}\beta_i(\theta) = 0, \quad \nu = 1, \ldots, r-1, \tag{6}$$

$$\frac{\theta^\nu}{\nu!} - \sum_{i=1}^{m} \frac{c_i^{\nu-1}}{(\nu-1)!}\beta_i(\theta) = 0, \quad \nu = r, \ldots, p. \tag{7}$$

Corollary 1. *The uniform order of convergence for a multivalue collocation method* (4) *is* $m + r - 1$.

Theorem 2. *An A-stable multivalue collocation method* (4) *fulfills the constraint* $r \leq m + 1$.

Theorem 3. *The order conditions in* (5)–(7) *imply:*

$$\alpha_j(0) = \delta_{j1},\ \alpha_j^{(\nu)}(0) = \delta_{j,\nu+1}, \quad j = 1, 2, ..., r, \quad \nu = 1, 2, ..., r-1, \tag{8}$$

$$\beta_j(0) = \beta_j^{(\nu)}(0) = 0, \quad j = 1, 2, ..., m, \quad \nu = 1, 2, ..., r-1, \tag{9}$$

$$\alpha_j'(c_i) = 0, \quad i = 1, 2, ..., r, \quad j = 1, 2, ..., m, \tag{10}$$

$$\beta_j'(c_i) = \delta_{ij}, \quad i, j = 1, 2, ..., m, \tag{11}$$

being δ_{ij} *the usual Kronecker delta.*

Proof. The conditions (9) follow immediately by substituting $\theta = 0$ in (7) and in its derivatives. The conditions (8) follow from (5)–(6) and (9), substituting $\theta = 0$. To show (11), we differentiate (7) and replace $\theta = c_i, i = 1, ..., m$, while (10) is derived by the differentiation of (6), putting $\theta = c_i, i = 1, ..., m$, and (11).

3 Construction of Almost Collocation Multivalue Methods with Diagonal Coefficient Matrix

The computational cost of the method (2) is strictly connected to the structure of the matrix **A**. In order to reduce this cost, we want to construct a multivalue method with a diagonal matrix **A**, so we have to determine the functional basis $\{\beta_j(\theta), j = 1, ..., m\}$ such that $\beta_j(c_i) = 0$ for $i \neq j$. In this way, we can not impose all the collocation conditions, but we have to relax some of them.

The following theorem holds.

Theorem 4. *A multivalue collocation method* (4) *has a diagonal coefficient matrix* **A** *and order* $p = r - 1$ *if*

$$\beta_j(\theta) = \omega_j(\theta) \prod_{k=1, k\neq j}^{m} (\theta - c_k), \quad j = 1, ..., m, \tag{12}$$

where $\omega_j(\theta)$ *is a polynomial of degree* $r - m + 1$*:*

$$\omega_j(\theta) = \sum_{k=0}^{r-m+1} \mu_k^{(j)} \theta^k, \tag{13}$$

and

$$\alpha_1(\theta) = 1 \tag{14}$$

$$\frac{\theta^\nu}{\nu!} - \alpha_{\nu+1}(\theta) - \sum_{i=1}^{m} \frac{c_i^{\nu-1}}{(\nu-1)!} \omega_i(\theta) \prod_{k=1, k\neq i}^{m} (\theta - c_k) = 0, \ \nu = 1, ..., r-1. \tag{15}$$

Proof. We want $\mathbf{A}$ to be diagonal, so we have to impose that $\beta_j(c_i) = 0$ for $i \neq j$. If we substitute c_i in (12), we obtain:

$$\beta_j(c_i) = \omega_j(c_i) \prod_{k=1, k\neq j}^{m} (c_i - c_k) = 0, \quad j = 1, ..., m, \tag{16}$$

so (12) is proved. Moreover, (14)–(15) are obtained by replacing (12) in (5)–(6).

We observe that $\beta_j(\theta), j = 1, ..., m$, are polynomial of degree r and conditions (14)–(15) permit to compute the functions $\alpha_i(\theta), i = 1, ..., r$, from $\beta_j(\theta)$. In the following we will fix $r = m + 1$. The parameters $\mu_k^{(j)}$ are free parameters which can be chosen in order to obtain A-stable methods. In searching for A-stable formulae, we have to analyze the properties of the stability matrix:

$$\mathbf{M}(z) = \mathbf{V} + z\mathbf{B}(I - z\mathbf{A})^{-1}\mathbf{U}, \tag{17}$$

where I is the identity matrix in $\mathbb{R}^{m\times m}$. In particular, we are interested in the computation of the roots of the stability function of the method:

$$p(\omega, z) = det(\omega I - \mathbf{M}(z)). \tag{18}$$

This roots have to be in the unit circle for all $z \in \mathbb{C}$ such that $Re(z) \leq 0$. By the maximul principle, that will happen if the denominator of $p(\omega, z)$ does not have poles in the negative half plane $\mathbb{C}_-$ and if the roots of the $p(\omega, iy)|$ are in the unit circle for all $y \in \mathbb{R}$. The last condition can be verified using the following Schur criterion.

Criterion 5. *Consider the polynomial [47]*

$$\phi(\omega) = c_k\omega^k + c_{k-1}\omega^{k-1} + ... + c_1\omega + c_0, \tag{19}$$

where c_i are complex coefficient, $c_k \neq 0$ and $c_0 \neq 0$, $\phi(\omega)$ is said to be a Schur polynomial if all its roots $\omega_i, i = 1, 2, ..., k$ are inside the unit circle. Define

$$\hat{\phi}(\omega) = \bar{c}_0\omega^k + \bar{c}_1\omega^{k-1} + ... + \bar{c}_{k-1}\omega + \bar{c}_k, \tag{20}$$

where $\bar{c}_i$ is the complex conjugate of c_i. Define also the polynomial

$$\phi_1(\omega) = \frac{1}{\omega}\left(\hat{\phi}(0)\phi(\omega) - \phi(0)\hat{\phi}(\omega)\right) \tag{21}$$

of degree at most $k - 1$. The following theorem holds.

Theorem 6. *(Schur). $\phi(\omega)$ is a Schur polynomial if and only if*

$$\left|\hat{\phi}(0)\right| > |\phi(0)| \tag{22}$$

and $\phi_1(\omega)$ is a Schur polynomial [47].

4 Examples of Methods

In this section we present examples of A-stable methods with two and three stages.

4.1 Two-Stage Methods

According to Theorem 4, we fix $\beta_j(\theta)$ as in (12) with $m = 2$ and $r = 3$, so $\omega_j(\theta)$ are polynomial of degree 2 of the form:

$$\omega_j(\theta) = \mu_0^{(j)} + \mu_1^{(j)}\theta + \mu_2^{(j)}\theta^2, \tag{23}$$

and the collocation polynomial is:

$$\begin{aligned} P_n(t_n + \vartheta h) = y_1^{[n]} + \alpha_2(\vartheta) y_2^{[n]} + \alpha_3(\vartheta) y_3^{[n]} + h\,(\beta_1(\vartheta) f(P(t_n + c_1 h)) \\ + \beta_2(\vartheta) f(P(t_n + c_2 h)))\,. \end{aligned}$$

We choose the values for the parameters $\mu_k^{(j)}$ in (23) by imposing the condition (7) for $\nu = r$ and by performing the Schur analysis of the characteristic polynomial of the stability matrix corresponding to the Butcher tableau:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}\right] = \left[\begin{array}{cc|ccc} \beta_1(c_1) & 0 & 1 & \alpha_2(c_1) & \alpha_3(c_1) \\ 0 & \beta_2(c_2) & 1 & \alpha_2(c_2) & \alpha_3(c_2) \\ \hline \beta_1(1) & \beta_2(1) & 1 & \alpha_2(1) & \alpha_3(1) \\ \beta_1'(1) & \beta_2'(1) & 0 & \alpha_2'(1) & \alpha_3'(1) \\ \beta_1''(1) & \beta_2''(1) & 0 & \alpha_2''(1) & \alpha_3''(1) \end{array}\right] \tag{24}$$

We obtain

$$\begin{gathered} \mu_0^{(1)} = 0, \quad \mu_1^{(1)} = \frac{1}{3(c_1 - c_2)}, \quad \mu_2^{(1)} = 0, \\ \mu_0^{(2)} = 0, \quad \mu_1^{(2)} = -\frac{c_1}{3(c_1 - c_2)c_2}, \quad \mu_2^{(2)} = \frac{1}{3c_2^2}, \end{gathered}$$

so

$$\begin{gathered} \alpha_2(\vartheta) = \frac{-\vartheta^3 + \vartheta^2(c_1 + c_2) + \vartheta(2c_2^2 - c_1 c_2)}{3c_2^2}, \\ \alpha_3(\vartheta) = \frac{-2\vartheta^3 + \vartheta^2(2c_1 + 3c_2) - 2c_1 c_2 \vartheta}{6c_2}, \\ \beta_1(\vartheta) = \frac{\vartheta(\vartheta - c_2)}{3(c_1 - c_2)}, \quad \beta_2(\vartheta) = \frac{(\vartheta^2(c_2 - c_1) + c_1 c_2 \vartheta)(c_1 - \vartheta)}{3c_2^2(c_1 - c_2)}. \end{gathered} \tag{25}$$

These methods have order 3. Figure 1 shows the region of A-stability in the (c_1, c_2) plane obtained from the Schur analysis of the method (24)–(25).

As an example, we chose $c_1 = 3$ and $c_2 = 29/10$, obtaining:

$$\begin{gathered} \alpha_2(\vartheta) = \vartheta\left(-\frac{25}{216}\vartheta^2 + \frac{\vartheta}{432} + \frac{359}{360}\right), \quad \alpha_3(\vartheta) = \vartheta\left(-\frac{5}{36}\vartheta^2 + \frac{61}{360}\vartheta + \frac{119}{300}\right), \\ \beta_1(\vartheta) = \frac{2}{3}\vartheta\left(\vartheta - \frac{6}{5}\right), \quad \beta_2(\vartheta) = \vartheta\left(\frac{25}{216}\vartheta^2 - \frac{289}{432}\vartheta + \frac{289}{360}\right). \end{gathered}$$

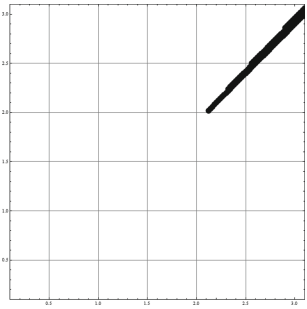

Fig. 1. Region of A-stability in the (c_1, c_2) plane.

which is the continuous C^2 extension of uniform order $p = 3$ of the A-stable multivalue method:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & \frac{3}{2} \\ 0 & \frac{29}{30} & 1 & \frac{29}{15} & \frac{841}{600} \\ \hline \frac{209}{15} & -\frac{37520}{2523} & 1 & \frac{24446}{12615} & \frac{1589}{870} \\ -\frac{62}{5} & \frac{11260}{841} & 0 & \frac{47}{4205} & -\frac{91}{145} \\ -\frac{91}{15} & \frac{5660}{841} & 0 & -\frac{8369}{12165} & -\frac{46}{145} \end{array}\right]$$

4.2 Three-Stage Multivalue Almost Collocation Method

Now we fix $m = 3$ and $r = 4$, so:

$$P_n(t_n + \vartheta h) = y_1^{[n]} + \alpha_2(\vartheta) y_2^{[n]} + \alpha_3(\vartheta) y_3^{[n]} + \alpha_4(\vartheta) y_4^{[n]} \\ + h\left(\beta_1(\vartheta) f(P(t_n + c_1 h)) + \beta_2(\vartheta) f(P(t_n + c_2 h)) + \beta_3(\vartheta) f(P(t_n + c_3 h))\right).$$

According to Theorem 4 we obtain a method of order $p = 3$, by fixing $\beta_j(\theta)$ as in (12) and finding the parameters $\mu_k^{(j)}$ such that the method is A-stable, so we perform Schur analysis of the characteristic polynomial of the stability matrix corresponding to the Butcher tableau:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}\right] = \left[\begin{array}{ccc|ccc} \beta_1(c_1) & 0 & 0 & 1 & \alpha_2(c_1) & \alpha_3(c_1) & \alpha_4(c_1) \\ 0 & \beta_2(c_2) & 0 & 1 & \alpha_2(c_2) & \alpha_3(c_2) & \alpha_4(c_2) \\ 0 & 0 & \beta_3(c_3) & 1 & \alpha_2(c_3) & \alpha_3(c_3) & \alpha_4(c_3) \\ \hline \beta_1(1) & \beta_2(1) & \beta_3(1) & 1 & \alpha_2(1) & \alpha_3(1) & \alpha_4(1) \\ \beta_1'(1) & \beta_2'(1) & \beta_3'(1) & 0 & \alpha_2'(1) & \alpha_3'(1) & \alpha_4'(1) \\ \beta_1''(1) & \beta_2''(1) & \beta_3''(1) & 0 & \alpha_2''(1) & \alpha_3''(1) & \alpha_4''(1) \\ \beta_1'''(1) & \beta_2'''(1) & \beta_3'''(1) & 0 & \alpha_2'''(1) & \alpha_3'''(1) & \alpha_4'''(1) \end{array}\right]$$

We find:

$$\begin{aligned}
\alpha_2(\vartheta) &= \left(-\frac{2\vartheta^4}{c_3^2}+\frac{2\vartheta}{3}+\frac{2\vartheta^3}{c_3^2}(c_1+c_2+c_3)\right)(5c_1^2+2c_2^2-7c_3^2)\\
&\quad+\left(-\frac{2\vartheta^2}{c_3^2}(c_1c_2+c_1c_3+c_2c_3)+\frac{2\vartheta}{c_3^2}c_1c_2c_3\right)(5c_1^2+2c_2^2-7c_3^2),\\
\alpha_3(\vartheta) &= \left(\frac{2\vartheta^4}{c_3}-\frac{2\vartheta^3}{c_3}(c_1+c_2+c_3)\right)(5c_1(c_3-c_1)+2c_2(c_3-c_2))\\
&\quad+\left(\frac{2\vartheta^2}{c_3}(c_1c_2+c_1c_3+c_2c_3)-\frac{2\vartheta}{c_3}c_1c_2\right)(5c_1(c_3-c_1)+2c_2(c_3-c_2))\\
&\quad+\frac{\vartheta^2}{6c_3}c_2(c_1c_2+c_1c_3+c_2c_3),\quad \alpha_4(\theta)=0,\\
\beta_1(\vartheta) &= \omega_1(\vartheta)(\vartheta-c_2)(\vartheta-c_3),\quad \beta_2(\vartheta)=\omega_2(\vartheta)(\vartheta-c_1)(\vartheta-c_3),\\
\beta_3(\vartheta) &= \omega_3(\vartheta)(\vartheta-c_1)(\vartheta-c_2),
\end{aligned}$$

where

$$\begin{aligned}
\omega_1(\vartheta) &= \frac{\vartheta(30(\vartheta-c_1)(c_1c_2+c_1c_3-c_2c_3-c_1^2)+1)}{3(c_1-c_2)(c_1-c_3)},\\
\omega_2(\vartheta) &= \frac{\vartheta(12(\vartheta-c_2)(c_1c_2-c_1c_3+c_2c_3-c_2^2)+1)}{3(c_2-c_1)(c_2-c_3)},\\
\omega_3(\vartheta) &= \frac{\vartheta((\vartheta(30c_1^2+12c_2^2)-30c_1^2c_3-12c_2^2c_3)(c_1c_3-c_1c_2+c_2c_3-c_3^2)-c_3^2)}{3c_3^2(c_1-c_3)(c_3-c_2)}.
\end{aligned}$$

For those polynomials, we perform again Schur analysis fixing one value for time of the abscissa coefficients. So Fig. 2 shows the regions of A-stability in the (c_2, c_3) plane for $c_1 = 9/5$, in the (c_1, c_3) plane for $c_2 = 8/5$ and in the (c_1, c_2) plane for $c_3 = 17/10$, respectively.

As an example, we chose $c_1 = 9/5$, $c_2 = 8/5$ and $c_3 = 17/10$, obtaining:

$$\begin{aligned}
\alpha_2(\vartheta) &= \vartheta\left(-\frac{218}{289}\vartheta^3+\frac{327}{85}\vartheta^2-\frac{47197}{7225}\vartheta+\frac{27794}{6375}\right),\\
\alpha_3(\vartheta) &= \vartheta\left(-\frac{58}{85}\vartheta^3+\frac{87}{25}\vartheta^2-\frac{73217}{12750}\vartheta+\frac{2088}{625}\right),\quad \alpha_4(\vartheta)=0,\\
\beta_1(\vartheta) &= \vartheta\left(-10\vartheta^3+\frac{203}{3}\vartheta^2-\frac{708}{5}\vartheta+\frac{7072}{75}\right),\\
\beta_2(\vartheta) &= \vartheta\left(-4\vartheta^3+\frac{556}{15}\vartheta^2-\frac{6973}{75}\vartheta+\frac{8823}{125}\right),\\
\beta_3(\vartheta) &= 8\vartheta\left(\frac{533}{289}\vartheta^3-\frac{3461}{255}\vartheta^2+\frac{653246}{21675}\vartheta-\frac{44688}{2125}\right),
\end{aligned}$$

which is the continuous C^2 extension of uniform order $p = 3$ of the A-stable general linear method:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}\right] = \left[\begin{array}{ccc|ccc} \frac{3}{5} & 0 & 0 & 1 & \frac{6}{5} & \frac{27}{50} & 0 \\ 0 & \frac{8}{15} & 0 & 1 & \frac{16}{15} & \frac{32}{75} & 0 \\ 0 & 0 & \frac{17}{30} & 1 & \frac{17}{15} & \frac{289}{600} & 0 \\ \hline \frac{259}{25} & \frac{4004}{375} & -\frac{757088}{36125} & 1 & \frac{99718}{108375} & \frac{25241}{63750} & 0 \\ -\frac{1943}{75} & -\frac{7561}{375} & \frac{5120776}{108375} & 0 & -\frac{19637}{108375} & -\frac{13822}{31875} & 0 \\ \frac{14}{5} & -\frac{866}{75} & \frac{168656}{21675} & 0 & \frac{6976}{7225} & \frac{7693}{6375} & 0 \\ 166 & \frac{632}{5} & -\frac{429712}{1445} & 0 & \frac{7194}{1445} & \frac{1914}{425} & 0 \end{array}\right]$$

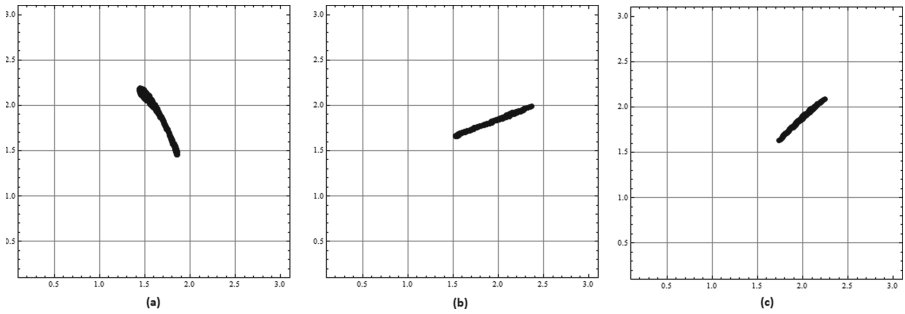

Fig. 2. Region of A-stability: **(a)** in the (c_2, c_3) plane for $c_1 = 9/5$; **(b)** in the (c_1, c_3) plane for $c_2 = 8/5$; **(c)** in the (c_1, c_2) plane for $c_3 = 17/10$.

5 Numerical Results

In this section we present numerical results for the methods introduced previously. In particular we denote with:

- GLM2: the method in Sect. 4.1
- GLM3: the method in Sect. 4.2
- RK2: the two stages Gaussian Runge-Kutta method:

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

The methods GLM2 and GLM3 have uniform order $p = 3$, while RK2 has order 4 and uniform order 2, therefore it suffers from order reduction when the problem is stiff.

We consider the Prothero-Robinson problem:

$$\begin{cases} y'(t) = \lambda(y(t) - sin(t)) + cos(t), \, t \in [0, 10], \\ y(t_0) = y_0, \end{cases} \tag{26}$$

with $Re(\lambda) < 0$ which is stiff when $\lambda \ll 0$.

Table 1 and 2 show the error in the final step point for different values of the step size and the experimental order of methods GLM2, GLM3 and RK2, respectively, for different values of λ in problem (26).

Table 1. Absolute errors (in the final step point) and effective orders of convergence for problem (26) with $\lambda = -10^3$.

h	GLM2		GLM3		RK2	
	Error	p	Error	p	Error	p
1/10	$4.9008\ 10^{-5}$		$4.1930\ 10^{-6}$		$1.77\ 10^{-4}$	
1/20	$3.0606\ 10^{-6}$	4.0011	$2.6733\ 10^{-7}$	3.9713	$1.32\ 10^{-5}$	3.75
1/40	$1.9182\ 10^{-7}$	3.9960	$1.7166\ 10^{-8}$	3.9610	$7.82\ 10^{-7}$	4.08
1/80	$1.2089\ 10^{-8}$	3.9880	$1.1240\ 10^{-9}$	3.9328	$4.78\ 10^{-8}$	4.03

Table 2. Absolute errors (in the final step point) and effective orders of convergence for problem (26) with $\lambda = -10^6$.

h	GLM2		GLM3		RK2	
	Error	p	Error	p	Error	p
1/10	$4.8836\ 10^{-5}$		$4.1468\ 10^{-6}$		$1.52\ 10^{-4}$	
1/20	$3.0403\ 10^{-6}$	4.0057	$2.6123\ 10^{-7}$	3.9886	$3.84\ 10^{-5}$	1.98
1/40	$1.8934\ 10^{-7}$	4.0052	$1.6450\ 10^{-8}$	3.9892	$9.99\ 10^{-6}$	1.94
1/80	$1.1849\ 10^{-8}$	3.9981	$1.0133\ 10^{-9}$	4.0210	$2.78\ 10^{-6}$	1.85

We can notice that the experimental order is consistent with the theoretical one and, even in the case of stiffness, for GLM2 and GLM3.

6 Conclusions

In this paper multivalue collocation methods with diagonal coefficient matrix have been presented. We prove that these methods have at least order $p = r - 1$ and we have constructed A-stable methods with two and three stages with order $p = 3$. Thanks to the structure of the coefficient matrix, those methods can be easily parallelized, so the computational effort can be reduced. In the future we aim to construct such types of methods for different operators such as stochastic differential equations [15,21,31], fractional differential equations [2,10,13,16], partial differential equations [1,11,14,20,30,32,35,38,40], Volterra integral equations [8,9,12,17,23], second order problems [26,37], oscillatory problems [19,22,24,33,36,53], as well as to the development of algebraically stable high order collocation based multivalue methods [18,29].

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