

Mathematical Modelling and Identification of a Quadrotor

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Abstract. Motivated by the important growth of VTOL vehicles research such as quadrotors and to a small extent autonomous flight, a quadrotor dynamical model is presented in this work. The purpose of this study is to get a better understanding of its flight dynamics. It is an underactuated system. So, a simplified and clear model is needed to implement controllers on these kind of unmanned aerial systems. In addition, a computational tool is used for validation purposes. For future works embedded or intelligent control systems can be developed to control them. Gyroscopic and some aerodynamics effects are neglected.

Keywords: Quadrotor · VTOL · Flight dynamics · UAV

1 Introduction

Humanity has been having a significant development increase that centuries ago it would not have been possible to believe that the first manned airplane had been built and flown by the Wright brothers in the early 1900s. Aircrafts are not the only ones with flight capacities. There are other flying vehicles like drones or unmanned aerial vehicles (UAVs) which can do so. Hence, there have been many researchers and engineers from different areas interested in developing aerial vehicles without the influence of humans. Several engineering areas such as aerodynamics, control, embedded electronics are associated to this type of systems. These type of vehicles can have small designs that favor their abilities for carrying or payload. The term *drones* has been used because of the autonomy constraints they could have. This is the reason why embedded and guidance control systems are applied to drones which permit autonomous flight tasks. One of the categories of UAVs is the multirotor which has the possibility of vertical takeoff and landing (VTOL). Nowadays, unmanned aerial vehicles

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play a very meaningful role in the current aerospace industry. They can provide different autonomous flight applications such as environmental research, rescue, traffic monitoring, agricultural inspections, image and video, scientific research, inspections of places with very difficult access and even more recently, home delivery of products. Therefore, it is important to remark that their uses are not just limited to dangerous roles. Due to the great progress of technology, UAVs have been evolving up to autonomous systems. They are capable of running by themselves without any kind of interventions from humans and with a pre-determined flight mission. Therefore, this fact has drawn the attention to research more in-depth about autonomous aircrafts [3,10,16,17].

2 Reference Frames

It is necessary to have at least one reference frame to describe any position or motion. The use of additional reference frames will make the derivation of the equations of motions easier. When using multiple reference frames there is one important issue which is related to the transformation of vector coordinates from one frame to another. The rotation matrices used are based on the Euler angles. A reference fixed frame is applied to determine distance and direction. A coordinate system is used to represent measurements in a frame. In flight dynamics there are two reference frames clearly defined, the Earth Fixed Frame and Body Fixed Frame [5,12,14,17]. The E-frame is chosen as the inertial frame. $(0_E\ X_E\ Y_E\ Z_E)$. The origin is at O_E. In this reference frame the linear position (ξ^E) and angular position (Θ^E) of the quadrotor are defined. The other reference frame required is the body frame $(0_B\ X_B\ Y_B\ Z_B)$ which is attached to the quadrotor body. The origin is at the vehicle's reference point O_B. In this B-frame the linear velocity, the angular velocity and the forces and torques are defined.

3 Quadrotor Assumptions for Modeling

Considering that computation of any model is just an approximation to real conditions found in the real world, some assumptions need to be made for simulation purposes:

- The quadrotor is treated as a rigid structure.
- The propellers are rigid. Aerodynamics effects such as blade flapping are neglected.
- Gyroscopic effects and aerodynamic torques can be ignored.
- The quadrotor structure is treated as symmetrical.
- Wind disturbances are ignored.

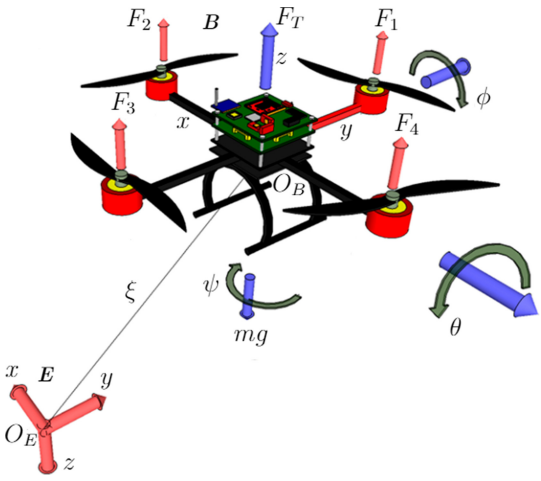

Fig. 1. Reference systems

4 Quadrotor System Variables

The quadrotor motion has six degrees of freedom (6DOF) which are defined as follows: ξ = (x, y, z) that represents the linear position of the quadrotor and $\Theta = (\theta, \phi, \psi)$ is the attitude or orientation. These are also known as the Euler angles *pitch, roll* and *yaw*. Therefore, if $\Theta = (\theta, \phi, \psi)$ and ξ = (x, y, z), so the general position vector Φ is (Fig. 1):

$$\Phi = \begin{bmatrix} \xi \\ \Theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \\ \psi \end{bmatrix} \tag{1}$$

Let v and ω be the quadrotor linear and angular velocities $v = (u, v, w)$ $\omega = (p, q, r)$. Therefore, the linear and angular velocities are:

$$v = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \tag{2}$$

4.1 Quadrotor Kinematics

To be able to transform the vectors from the E-frame to the B-frame a direction cosine matrix is required. If the rotations are done first around x

axis then around y and the final one around z axis, the rotation matrix is: $R_\Upsilon = R(\phi, \theta, \psi) = R_x(\phi)R_y(\theta)R_z(\psi)$ [14,15,17,18].

Where:

Roll rate: $p = \dot{\phi} - \dot{\psi}\ sin\ \theta$
Pitch rate: $q = \dot{\theta}\ cos\ \phi + \dot{\psi}\ sin\ \phi\ cos\ \theta$
Yaw rate: $r = \dot{\psi}\ cos\ \phi cos\ \theta - \dot{\theta}\ sin\ \phi$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix} \tag{3}$$

$$R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \tag{4}$$

$$R_z(\psi) = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Then, the complete rotation matrix is the product of the three rotation matrices:

$$R_{\Upsilon_B^E} = \begin{bmatrix} cos\theta cos\psi & cos\psi sin\theta sin\phi - sin\psi cos\phi & sin\phi sin\psi + cos\phi cos\psi sin\theta \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & sin\psi sin\theta cos\phi - cos\psi sin\phi \\ -sin\theta & cos\theta sin\phi & cos\phi cos\theta \end{bmatrix} \tag{6}$$

The relationship between the angular velocity $\dot{\omega}$ in (E-frame) and the angular velocity in (B-frame) is given by: $\dot{\omega} = T_\chi\ \omega$. The transformation matrix T_χ is found by computing the attitude rates $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ and body rates (p, q, r) as follows:

$$v^E = \dot{\xi}^E = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = R_\Upsilon v^B \tag{7}$$

$\Rightarrow$

$$v^E = R_\Upsilon = \begin{bmatrix} cos\theta cos\psi & cos\psi sin\theta sin\phi - sin\psi cos\phi & sin\phi sin\psi + cos\phi cos\psi sin\theta \\ sin\psi cos\theta & cos\psi cos\phi + sin\psi sin\theta sin\phi & sin\psi sin\theta cos\phi - cos\psi sin\phi \\ -sin\theta & cos\theta sin\phi & cos\phi cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{8}$$

The angular body rates p, q, r are shown in Fig. 2. First, *roll* rotates around the angle ϕ and angular velocity $\dot{\phi}$. Then, *pitch* which rotates around the θ angle with $\dot{\theta}$ angular velocity. Likewise, *yaw* rotates through angle ψ with angular velocity $\dot{\psi}$. Hence, the relationship between the aerial vehicle body rates and attitude rates is given as:

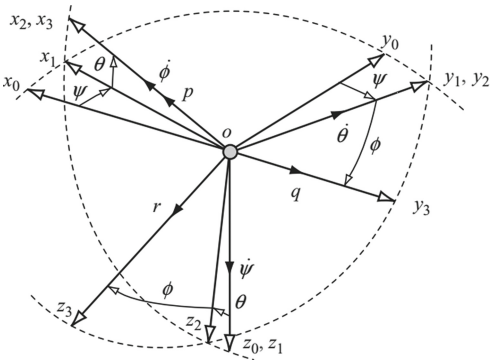

Fig. 2. Angular velocities transformation

Roll rate:

$$p = \dot{\phi} - \dot{\psi}\ sin\ \theta \tag{9}$$

Pitch rate:

$$q = \dot{\theta}\ cos\ \phi + \dot{\psi} sin\phi\ cos\theta \tag{10}$$

Yaw rate:

$$r = \dot{\psi}\ cos\phi + cos\theta - \dot{\theta} sin\ \phi \tag{11}$$

When the quadrotor Euler angles (ϕ, θ, ψ) are considered small $(\phi, \theta, \psi \cong 0)$ the body rates equations can be taken as: [4,17,18].

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{12}$$

Since $[x,\ y,\ z,\ \phi,\ \theta,\ \psi]^T$ is the vector that contains the linear and angular position of the quadrotor in the E-frame and $[u,\ v,\ w,\ p,\ q,\ r]^T$ is the vector which contains the linear and angular velocities in the B-frame, then these two reference frames are related as follows:

$$v = R_{\Upsilon} \cdot v_B \tag{13}$$

$$\omega = T_{\chi} \cdot \omega_B \tag{14}$$

Finally, the kinematic model of a quadrotor is:

$$\begin{cases} \dot{x} &= w[sin(\phi)sin(\psi) + cos(\phi)cos(\psi)sin(\theta)] - v[cos(\phi)sin(\psi) - cos(\phi)sin(\psi)sin(\theta)] + u[cos(\psi)cos(\theta)] \\ \dot{y} &= v[cos(\phi)cos(\psi) + sin(\phi)sin(\psi)sin(\theta)] - w[cos(\psi)sin(\phi) - cos(\psi)sin(\psi)sin(\theta)] + u[cos(\theta)sin(\psi)] \\ \dot{z} &= w[cos(\phi)cos(\theta)] - u[sin(\theta)] + v[cos(\theta)sin(\phi)] \\ \dot{\phi} &= p + r[cos(\phi)tan(\theta)] + q[sin(\phi)tan(\theta)] \\ \dot{\theta} &= qcos(\phi) - rsin(\phi) \\ \dot{\psi} &= q\frac{sin(\phi)}{cos(\theta)} + r\frac{cos(\phi)}{cos(\theta)} \end{cases} \tag{15}$$

4.2 Quadrotor Dynamics

The second Newton's law states that a forced applied in the E reference frame is equal to the product of the vehicle' s mass and its acceleration. i.e, *Force* = *mass* × *acceleration*. Therefore, it can be inferred that:

$$F = m\left(\frac{dv}{dt}\right) = m\left(\frac{d^2\xi}{dt^2}\right) \tag{16}$$

Force = *mass* × *acceleration* Hence,

$$F = m\left(\frac{dv}{dt}\right) = m\left(\frac{d^2\xi}{dt^2}\right) \tag{17}$$

From Eq. 1:

$$\left(\frac{d^2\xi^E}{dt^2}\right) = \ddot{\xi^E} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \tag{18}$$

The total force exerted on the quadrotor is given by the following matrix relation:

$$m(\omega^B \times \ v^B + \dot{v}^B) = F^B \tag{19}$$

The total force exerted on the quadrotor is given by the following matrix relation:

$$m(\omega^B \times \ v^B + \dot{v}^B) = F^B \tag{20}$$

Where × is the cross product, m is the quadcopter mass and $F^B = [F_x \ F_y \ F_z]^T$ which is the total force. So, a skew symmetric matrix is got and expressed as: [17,18].

$$\omega^B \times v^B = \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} \tag{21}$$

And if the differential of the linear velocity is:

$$v = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} \tag{22}$$

Therefore, the following equation is obtained:

$$F^B = m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} \tag{23}$$

If the column vector $F^B = [F_x \ F_y \ F_z]^T$ is substituted and dividing by m, we get:

$$\frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} \tag{24}$$

Rewriting, the following is acquired:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} - \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} \tag{25}$$

Then, the total torque applied to the quadrotor is given by:

$$I\dot{\omega}^B + \omega^B \times (I\omega^B) + M_G = M^B \tag{26}$$

Where $\omega^B \times v^B$ is known as the Coriolis term, $[M_x \ M_y \ M_z] = M^B$ is the total torque, M_G gyroscopic effects due to rotor's inertia and I known as the inertia matrix [7,17,18].
A quadcopter is assumed as symmetric:

$$I_{xy} = I_{xz} = I_{yx} = I_{yz} = I_{zx} = I_{zy} = 0 \tag{27}$$

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{28}$$

The angular acceleration is given by:

$$\dot{\omega}^B = I^{-1}[T_\chi \tau^B - (\omega^B \times I\omega^B)] \tag{29}$$

Where τ^B is given as:

$$\tau^B = [\tau_r \ \tau_p \ \tau_y]^T \tag{30}$$

Therefore $T_\chi \tau^B$ is:

$$T_\chi \tau^B = \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} = I\dot{\omega}^B + [\omega^B \times (I\omega^B)] \tag{31}$$

The inverse of matrix 28 is:

$$I^{-1} = \begin{bmatrix} \frac{1}{I_{xx}} & 0 & 0 \\ 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix} \tag{32}$$

If the angular velocities in Eq. 2 are derived we obtain:

$$\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} \tag{33}$$

Now substituting in Eq. 29 results:

$$\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = I^{-1} \left[T_\chi \tau^B - \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) \right] \tag{34}$$

Simplifying:

$$\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = I^{-1} \left[\begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} - \begin{bmatrix} qrI_{zz} - rqI_{yy} \\ rpI_{xx} - prI_{zz} \\ pqI_{yy} - qpI_{xx} \end{bmatrix} \right] \tag{35}$$

Substituting I^{-1} in Eq. 35 we get:

$$\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{1}{I_{xx}} & 0 & 0 \\ 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix} \left[\begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} \begin{bmatrix} qrI_{zz} - rqI_{yy} \\ rpI_{xx} - prI_{zz} \\ pqI_{yy} - qpI_{xx} \end{bmatrix} \right] \tag{36}$$

Simplifying:

$$\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{\tau_r}{I_{xx}} \\ \frac{\tau_p}{I_{yy}} \\ \frac{\tau_y}{I_{zz}} \end{bmatrix} - \begin{bmatrix} I_{zz} - I_{yy} \frac{qr}{I_{xx}} \\ I_{xx} - I_{zz} \frac{pr}{I_{yy}} \\ I_{yy} - I_{xx} \frac{pq}{I_{zz}} \end{bmatrix} \tag{37}$$

4.3 Thrust and Moment

It is assumed that the torque and thrust caused by each rotor acts particularly in the z axis of the B-frame. Accordingly, the net propulsive force in the z^B direction is given by:

$$F_t = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix} \tag{38}$$

Moment results from the thrust action of each rotor around the center of mass that induces a pitch and roll motion. Furthermore, there is a reactive torque of the rotors on the vehicle which produces a yaw reply. The moment vector is therefore: [9,17].

$$M_t = \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} = \begin{bmatrix} lb(F_2 - F_4) \\ lb(F_3 - F_1) \\ (F_1 - F_2 + F_3 - F_4)d \end{bmatrix} \tag{39}$$

Where l is distance between any rotor and the center of the drone, b is the thrust factor and d is the drag factor [9,17,19].
The external forces used are based on the model found in [4,14]:

$$\begin{aligned} F_x &= -Wsin\theta + x = m(\dot{u} + qw - rv) \\ F_y &= Wcos\theta cos\psi + x = m(\dot{v} + ru - pw) \\ F_z &= Wcos\theta cos\psi + z = m(\dot{w} + pv - qu) \end{aligned} \tag{40}$$

$$\begin{aligned} M_x &= I_{xx}\dot{p} + (I_{zz} - I_{yy})qr \\ M_y &= I_{yy}\dot{q} + (I_{xx} - I_{zz})rp \\ M_z &= I_{zz}\dot{r} + (I_{yy} - I_{xx})pq \end{aligned} \tag{41}$$

The Matrix 42 shows the relationship between the the net torque that is performed on the vehicle and the angular acceleration. The total torque is the summation of the aerodynamic torque τ_a and the torque generated by the rotors [13].

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} Aero + \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} Quad = \begin{bmatrix} I_{xx}\dot{p} - (I_{yz}(q^2 r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) - (I_{yy} - I_{zz})qr \\ I_{yy}\dot{q} - I_{xz}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) - (I_{zz} - I_{xx})rp \\ I_{xx}\dot{p} - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) - (I_{yy} - I_{zz})qr \end{bmatrix} \tag{42}$$

Therefore, the quadrotor dynamic model in the B-frame is:

$$\begin{cases} F_x = m(\dot{u} + qw - rv) \\ F_y = m(\dot{v} - pw + ru) \\ F_z = m(\dot{w} + pv - qu) \\ M_x = I_{xx}\dot{p} + (I_{zz} - I_{yy})qr \\ M_y = I_{yy}\dot{q} + (I_{xx} - I_{zz})rp \\ M_z = I_{zz}\dot{r} + (I_{yy} - I_{xx})pq \end{cases} \tag{43}$$

The rotation of the propellers combined with the rotation of the body results in a gyroscopic torque which is given by:

$$M_G = J_m(\omega_B \times z^B)(\Omega_1 \ + \Omega_2 \ + \Omega_3 \ + \Omega_4) \tag{44}$$

Where Ω $(i = 1, 2, 3, 4)$ is propeller angular velocity, J_m is the inertia of each motor. Taking into an account that $\tau_B = [\tau_p \ \tau_r \ \tau_y]$ are the torques generated by the actuators action and $\tau_a = [\tau_{ax} \ \tau_{ay} \ \tau_{az}]$ are the aerodynamic torques, finally, the complete quadrotor dynamic model is:

$$\begin{cases} -mg[sin(\theta)] + F_{ax} = m(\dot{u} + qw - rv) \\ mg[cos(\theta)sin(\phi)] + F_{ay} = m(\dot{v} - pw + ru) \\ mg[cos(\theta)cos(\phi)] + F_{az} - ft = m(\dot{w} + pv - qu) \\ \tau_p + \tau_{ax} = \dot{p}I_{xx} - qrI_{yy} + qrI_{zz} \\ \tau_r + \tau_{ay} = \dot{q}I_{yy} + prI_{xx} - prI_{zz} \\ \tau_y + \tau_{az} = \dot{r}I_{zz} - pqI_{xx} + pqI_{yy} \end{cases} \tag{45}$$

$F^B = [F_x \ F_y \ F_z]$ [N], R_Υ is the rotation matrix and m [kg] is the mass of the quadrotor. So, it is possible to combine them as follows: [17,18].

$$F^B = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = mR_\Upsilon \frac{d}{dt}\ \dot{\xi} = mR_\Upsilon \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \tag{46}$$

They can be arranged as:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{m} R_\Upsilon^{-1} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{47}$$

Making substitutions the equation is simplified as:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{m} R_\Upsilon^{-1} \begin{bmatrix} -mgsin\theta \\ mgsin\phi cos\theta \\ mgcos\phi cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix} \tag{48}$$

Therefore:

$$\begin{cases} \ddot{x} = \frac{F_{total}}{m}[sin(\phi)sin(\psi) + cos(\phi)cos(\psi)sin(\theta)] \\ \ddot{y} = \frac{F_{total}}{m}[cos(\phi)sin(\psi)sin(\theta) - cos(\psi)sin(\phi)] \\ \ddot{z} = -g\frac{F_{total}}{m}[cos(\phi)cos(\theta)] \end{cases} \tag{49}$$

As stated earlier, for small angles of movement $[\dot{\phi}\ \dot{\theta}\ \dot{\psi}]^T = [p\ q\ r]^T$, hence, the dynamic model of quadrotor in the E-frame is:

$$\begin{cases} \ddot{x} = \frac{F_{total}}{m}[sin(\phi)sin(\psi) + cos(\phi)cos(\psi)sin(\theta)] \\ \ddot{y} = \frac{F_{total}}{m}[cos(\phi)sin(\psi)sin(\theta) - cos(\psi)sin(\phi)] \\ \ddot{z} = -g\frac{F_{total}}{m}[cos(\phi)cos(\theta)] \\ \ddot{\phi} = \frac{I_{yy}-I_{zz}}{I_{xx}}\dot{\theta}\dot{\psi} + \frac{\tau_x}{I_{xx}} \\ \ddot{\theta} = \frac{I_{zz}-I_{xx}}{I_{yy}}\dot{\phi}\dot{\psi} + \frac{\tau_x}{I_{yy}} \\ \ddot{\psi} = \frac{I_{xx}-I_{yy}}{I_{zz}}\dot{\phi}\dot{\theta} + \frac{\tau_x}{I_{zz}} \end{cases} \tag{50}$$

4.4 Motor and Propeller Dynamics

The following is the differential equation for a DC motor:

$$J_m\ \dot{\Omega}_m = -\frac{K_E\ K_q}{R}\ \Omega_m - \tau_l + \frac{K_q}{R}\ V \tag{51}$$

Where J_m is the motor inertia $[N\ m\ s^2]$, $\dot{\Omega}_m$ is the motor angular speed $[rad/s^2]$, τ_l is the load torque $[N\ m]$. There is an acceleration or deceleration of the angular speed of the motor Ω_m when the torques τ_m (motor torque) and τ_l are not the same. The motor torque τ_m is proportional to the current i and through the

torque constant K_q $[Amps/Nm]$, K_E is known as the voltage motor constant [rad s/V].
The power required to hover is given as:

$$P = TV_i \equiv TV_{rh} = T\sqrt{\frac{T}{2\rho A_b}} = \frac{T^{3/2}}{\sqrt{2\rho A_b}} \tag{52}$$

Where A_b is the area cleaned out by the rotor blades and ρ is the air density. The above expression can be related to the induced velocity v_{rh} in forward flight as:

$$v_i = \frac{v_{rh}^2}{\sqrt{(v_\infty cos\alpha)^2 + (v_i - v_\infty sin\alpha)^2}} \tag{53}$$

Where α is rotor angle of attack and v_∞ is velocity stream flow [8,11]. If Eq. 52 is substituted for v_{rh}, it results;

$$fom\ \eta_m \frac{\tau_m}{K_q} V = T\sqrt{\frac{T}{2\rho A_b}} \tag{54}$$

The rotor power in forward flight is given by:

$$P = T(v_\infty\ sin\ \alpha + v_i) \tag{55}$$

So, the ideal thrust T when a power input P can be calculated as follows:

$$T = \frac{P}{v_\infty\ sin\ \alpha + v_i} \tag{56}$$

Where the denominator is the airspeed across the rotors.
The motor torque is proportional to the thrust constant K_t:

$$\tau_m = K_t T \tag{57}$$

If Eq. 52 is substituted for τ_m, it gives:

$$fom\ \eta_m \frac{K_t T}{K_q} V = T\sqrt{\frac{T}{2\rho A_b}} \tag{58}$$

Torque τ_m is provided by each motor that is balanced by the drag torque. Accordingly, the torque acting on the propeller is:

$$\tau_m = J_m \dot{\Omega} + \tau_{drag} \tag{59}$$

Finally, the relationship between voltage and thrust is given by:

$$T = 2\rho A_b \left[\frac{fom\ \eta_m\ K_t}{K_q}\right]^2 V^2 \tag{60}$$

A quadrotor is controlled by providing four torques ($\tau_1\ \tau_2\ \tau_3\ \tau_4$) to the rotors which produce four thrust forces ($F_1\ F_2\ F_3\ F_4$) in the z axis. The net torques acting on the body can be calculated by using the inputs $U = (U_1\ U_2\ U_3\ U_4)$ that can be applied to control the quadrotor. As an aerodynamic consideration forces and torques are proportional to the squared propeller's speed [1].
Therefore the relationship between motions and propellers' squared speed is as follows:

$$\begin{aligned} F_T &= b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \\ \tau_{x(\theta)} &= bl(\Omega_2^2 - \Omega_4^2) \\ \tau_{y(\phi)} &= bl(\Omega_1^2 - \Omega_3^2) \\ \tau_{z(\psi)} &= K_{drag}(\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2) \end{aligned} \tag{61}$$

Where l [m] is the distance between any rotor and the center of the drone, b is the thrust factor [$N\ s^2$], K_{drag} is a drag constant [$N\ m\ s^2$] and Ω_i [$rad\ s^{-1}$] is the propeller angular acceleration [7,17,18]. Then the dynamic model of the quadrotor is:

$$\begin{cases} -mg[sin(\theta)] + F_{ax} = m(\dot{u} + qw - rv) \\ mg[cos(\theta)sin(\phi)] + F_{ay} = m(\dot{v} - pw + ru) \\ mg[cos(\theta)cos(\phi)] + F_{az} - b(\Omega_1^2 + \Omega_2^2 - \Omega_3^2 - \Omega_4^2) = m(\dot{w} + pv - qu) \\ bl(\Omega_3^2 - \Omega_1^2) + \tau_{ax} = \dot{p}I_x - qrI_y + qrI_z \\ bl(\Omega_4^2 - \Omega_2^2) + \tau_{ay} = \dot{q}I_y - prI_x + prI_z \\ K_{drag}(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) + \tau_{az} = \dot{r}I_z - pqI_x + pqI_y \end{cases} \tag{62}$$

The overall propeller's velocities Ω [$rad\ s^{-1}$] and propeller's velocities vector $\boldsymbol{\Omega}$ are defined as: [1].

$$\Omega = -\Omega_1 + \Omega_2 - \Omega_3 + \Omega_4 \tag{63}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} \tag{64}$$

The following equations system shows the relationship between the control inputs and the propellers' squared speed:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ bl(\Omega_4^2 - \Omega_2^2) \\ bl(\Omega_3^2 - \Omega_1^2) \\ d(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) \end{bmatrix} \tag{65}$$

Where l [m] is the distance from the center of the quadrotor and the propeller and [$U_1\ U_2\ U_3\ U_4$] are the propeller's speed control inputs. U_1 is the combination of thrust forces responsible for the quadrotor altitude z. U_2 is the differential

thrust between rotors 2 and 4 which generate the roll moment. U_3 is the thrust differential between rotors 1 and 3 that create the pitch moment. Finally, U_4 is the combination of the individual torques between the clockwise and counter-clockwise rotors in charge of generating yaw rotation. Consequently, U_1 generates the desired quadrotor altitude, U_2 and U_3 generate the respective roll and pitch angles whereas U_4 creates the yaw angle [1,2,6,7,18]. Simplifying and including U variables we get:

$$\mathbf{f(x,u)} = \begin{pmatrix} \dot{\phi} = \dot{\theta}\dot{\psi}\lambda_1 + \dot{\theta}\lambda_2\Omega_r + b_1U_2 \\ \dot{\theta} = \dot{\phi}\dot{\psi}\lambda_3 + \dot{\phi}\lambda_4\Omega_r + b_2U_3 \\ \dot{\psi} = \dot{\theta}\dot{\phi}\lambda_5 + b_3U_4 \\ \ddot{x} = \mu_x\dfrac{1}{m}U_1 \\ \ddot{y} = \mu_y\dfrac{1}{m}U_1 \\ \ddot{z} = g - (cos\phi cos\theta)\dfrac{1}{m}U_1 \end{pmatrix} \tag{66}$$

J_m $[N\ m/\ s^2]$ is the motor inertia.

5 Simulations and Results

Table 1 presents the selected variables for simulation purposes which are based on the AR drone. Since a complete math model is provided, other variables such as thrust and drag coefficients as well as motor inertia are involved in the flight control analysis. Therefore, after performing several control simulations and some gain adjustments, it can be seen that PD control has a good response getting a better and quick stabilization of the vehicle. Also, when the moments of inertia are of very small values, attitude stabilization is more difficult. Some overshoot or peaks are also noticed in attitude angles. For altitude, peaks are not really evident. The PID gains are computed as well. Additional corrections are required as some signal deviations from the desired references are observed. The mean of dynamics values are helpful for more appropriate results (Fig. 3).

Table 1. Quadrotor parameters (AR Drone)

Dynamic variables	Value
m	0.1 kg
I_x	0.45 kg · m^2
I_y	0.51 kg · m^2
I_z	0.95 kg · m^2
l	0.5 m

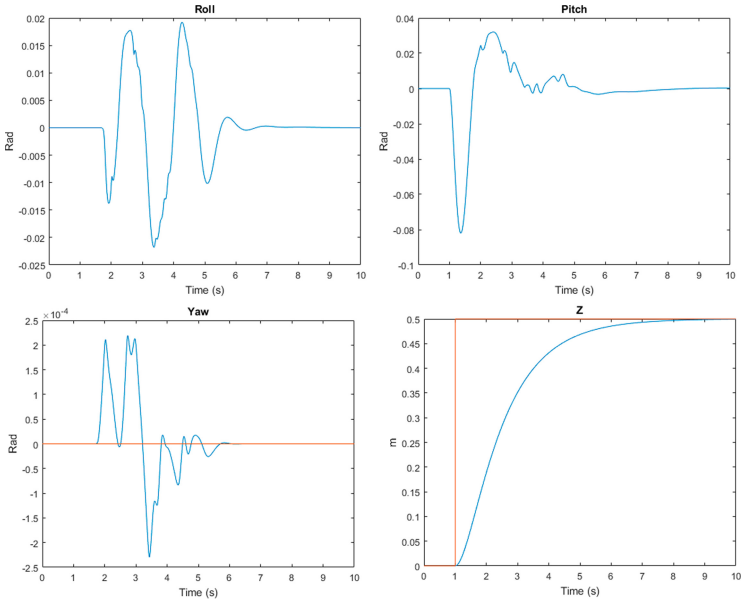

Fig. 3. Altitude/Attitude quadrotor performance

6 Conclusions

Vertical takeoff and landing aircrafts have a complex and highly non-linear dynamics. A complete model is given using the Newton-Euler approach. It could be used to predict changes in position and orientation of a quadcopter at a given point in time. It can be achieved by varying the rotors speed. Equations are inverted to find inputs needed for a certain position. Additionally, there are six degrees of freedom and just four inputs. Therefore, it is not appropriate and a simplified dynamics model would be required to be then implemented for control purposes. Also, there are some sensor measurements that are based on the E-frame and not on sensors frames. This model approach can be adapted to consider voltage inputs and not just U since there is an important association between thrust and voltage. It is necessary to adjust control gains. So, the smaller the moments of inertia and mass the more difficult for the platform to stabilize. State estimation would have a significant role because of an estimation of sensor measurements is necessary for autopilots design.

References

1. Bresciani, T.: Modelling, identification and control of a quadrotor helicopter. MSc theses (2008)
2. Cárdenas Ruiz, C.A.: Performance study of the flight control and path planning for a UAV type Quadrotor. Ingeniería Mecatrónica (2018)

3. Collazos, C., et al.: State estimation of a dehydration process by interval analysis. In: Figueroa-García, J.C., López-Santana, E.R., Rodriguez-Molano, J.I. (eds.) WEA 2018. CCIS, vol. 915, pp. 66–77. Springer, Cham (2018). https://doi.org/10.1007/978-3-030-00350-0_6
4. Cook, M.V.: Flight Dynamics Prıncıples. Elsevıer (2007)
5. Cork, L.R.: Aircraft dynamic navigation for unmanned aerial vehicles. Ph.D. thesis. Queensland University of Technology (2014)
6. Cowling, I.: Towards autonomy of a Quadrotor UAV (2008)
7. Habib, M.K., Abdelaal, W.G.A., Saad, M.S., et al.: Dynamic modeling and control of a Quadrotor using linear and nonlinear approaches (2014)
8. Huang, H., et al.: Aerodynamics and control of autonomous Quadrotor helicopters in aggressive maneuvering. In: IEEE International Conference on Robotics and Automation, ICRA 2009, pp. 3277–3282. IEEE (2009)
9. Ireland, M.L.: Investigations in multi-resolution modelling of the Quadrotor micro air vehicle. Ph.D. thesis. University of Glasgow (2014)
10. Jiménez-Cabas, J., et al.: Robust control of an evaporator through algebraic Riccati equations and d-K iteration. In: Misra, S., et al. (eds.) ICCSA 2019. LNCS, vol. 11620, pp. 731–742. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-24296-1_58
11. Leishman, G.J.: Principles of Helicopter Aerodynamics with CD Extra. Cambridge University Press, Cambridge (2006)
12. Lugo-Cárdenas, I., et al.: Dubins path generation for a fixed wing UAV. In: 2014 International Conference on Unmanned Aircraft Systems (ICUAS), pp. 339–346. IEEE (2014)
13. Malang, Y.: Design and Control of a Vertical Takeo and Landing Fixed-wing Unmanned Aerial Vehicle. Ph.D. thesis (2016)
14. Mulder, S.: Flight Dynamics (2007)
15. Swartling, J.O.: Circumnavigation with a group of Quadrotor helicopters (2014)
16. Pharpatara, P.: Trajectory planning for aerial vehicles with constraints. Theses. Université Paris-Saclay; Université d'Evry-Val-d'Essonne, September 2015. https://tel.archives-ouvertes.fr/tel-01206423
17. Poyi, G.T.: A novel approach to the control of quad-rotor helicopters using fuzzy-neural networks (2014)
18. Sabatino, F.: Quadrotor control: modeling, nonlinear control design, and simulation (2015)
19. Voos, H.: Nonlinear control of a Quadrotor micro-UAV using feedback linearization. In: International Conference on Mechatronics, ICM 2009, pp. 1–6. IEEE (2009)