Parameter Estimation Problems in Markov Random Processes

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Abstract. Problems of convergence and stability of Bayesian estimates in the identification of stochastic control systems are considered. The informational measure of the mismatch between the estimated distribution and the estimate is the main apparatus for establishing the fact of convergence. The choice of a priori distribution of parameters is not always obvious. The Kullback-Leibler information number is taken as such measure. The convergence of the estimates of the transition function of the process to the non-stationary transition function is established in this paper. The problem of synthesis of optimal strategies for dynamic systems in which there is no part of the main information needed for constructing the optimal control is also considered. It is assumed that the system contains at least one unknown parameter belonging to some parameter space. Therefore, the class of control systems considered in the article is the class of parametric adaptive systems.

Keywords: Bayesian probability theory · Parameter estimation · Kullback-Leibler information number

1 Introduction

Recently in connection with the presentation of increasingly higher requirements to control processes in various fields of technology there appeared a need in the management of such objects for which not only there is no adequate mathematical model but sometimes even general qualitative research is little studied. An emerging approach for new control problems is based on the idea of adapting a control system to the properties of particular objects to which it refers is known in advance.

Let the process $x_t \in X \subset R^m$ be observed. It is assumed that the conditional distribution $P_t(x_{t+1}|x^t, \theta)$ depends on a parameter $\theta \in \Theta$ the value of which is not known, but is constant throughout the experiment. This situation occurs, for example, when

$$x_{t+1} = f(x_t, \theta, \eta_t),$$

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where $\eta_t, t = 1, ...$ is a sequence of independent identically distributed random variables and η_t does not depend on $x_t \in X \subset R^m$.

In mathematical statistics there are many methods for parameter estimation. For example, the minimax approach [1], where statistical decisions are evaluated according to the "worst" possible values of θ or the Bayesian approach in which for an unknown parameter θ some distribution $\nu(d\theta)$ called a priori distribution is defined on $(\Theta, \mathcal{F})$.

In the paper we consider the Bayesian approach. Further we assume that $P_\theta(dx_t|x^t) = P_t(dx_t|x^t, \theta)$. Let $\nu(d\theta)$ belong to some family $\aleph$.

Assume that the family $\aleph$ is dominated by a measure $n(d\theta)$. A density $\nu(d\theta)$ with respect to $n(d\theta)$ is denoted by $\nu(d\theta)$. Suppose that $\varphi(\theta)$ is some measurable function of a parameter θ.

Definition 1. *The function $\nu_t(\theta|x^t)$ is called a posteriori distribution density of θ.*

Definition 2. *The value*

$$\bar{\varphi}(x^t) = \int_\Theta \varphi(\theta)\nu_t(\theta|x^t)n(d\theta)$$

is called the Bayesian estimate of the function $\varphi(\theta)$,

where

$$\nu_t(\theta|x^t) = \frac{P_\theta(x^t)\nu_0(\theta)}{\int\limits_\Theta P_\theta(x^t)\nu_0(\theta)n(d\theta)}. \tag{1}$$

We study the Bayesian estimation for the transition function $P_\theta(dx_t|x^t)$. The process x_t defined by a transition function $P_\theta(dx_t|x^t)$ can be interpreted as a partially observed controlled Markov process in phase space $\Theta \times X$ with a transition function for $\theta : \theta_t = \theta_{t+1}$. Let us use the well-known construction for constructing an auxiliary model of the process with complete information. As phase space we consider the space $\{X, \aleph\}$, where $\aleph$ is a distribution space on Θ. It is a Borel space. It follows from the fact that a set of probabilistic measures on the Borel space is a Borel space. Define a transition function in the space $\{x, \aleph\}$

$$P(x_{t+1}|x_t, \nu_t) = \int_\Theta P(x_{t+1}|x_t)\nu_t n(d\theta)$$

where the distribution density ν_{t+1} is determined by formula Eq. (1),

$$\nu_t(\theta) = \frac{P_\theta(x_{t+1}|x_t)}{\int\limits_\Theta P_\theta(x_{t+1}|x_t, \nu_0(\theta))}.$$

To specify a process we define an initial distribution $P_1(x_1)$ and a priori density $\nu_1(\theta) = \nu_1/dn$.

In this paper closed control systems are considered. Let us consider a linear difference equation of order n

$$x_{t+1} = \sum_{k=0}^{n-1} \alpha_k x_{t-k} + \sum_{k=0}^{m} \beta_k u_{t-k} + f_{t+1}, \tag{2}$$

where α_k, β_k are constant coefficients; f_t is an independent sequence of Gaussian quantities with zero mean and a variance σ^2, u_t is a control, $\beta_0 \neq 0$. Assume that coefficients α_k are unknown. Denote

$$w_t = \sum_{k=0}^{m} \beta_k u_{t-k}, \quad \theta^T = (\alpha_0, \alpha_1, ..., \alpha_{n-1}).$$

Then

$$x_{t+1} = \theta^T z_t + w_t + f_{t+1}, \tag{3}$$

where $z_t^T = (x_t, x_{t-1}, ..., x_{t-n+1})$. The transient function of process $P_\theta(x_{t+1}|z_t, w_t)$ defined by Eq. (3) has the form

$$P_\theta(x_{t+1}|z_t, w_t) = F(x_{t+1} - \theta^T z_t - w_t),$$

where $F(\cdot)$ is a distribution density of a random variable f_t.

2 Problem Statements

In real situations a transition function of the controlled Markov process depends on an unknown parameter. Generally speaking, for control purposes the knowledge of this parameter θ is not necessary, but the proximity of the distribution estimate to the true distribution is desirable. In order to formulate the concept of proximity of distribution functions we introduce a definition [2].

Definition 3. *A set of subsets A_i of a space Y is called a partition of the space Y if all subsets A_i are disjoint and their union equals Y.*

Then the proximity of two distribution functions P_1, P_2 is understood in the sense of a metric which can be determined by the following relation [3]

$$\rho(P_1; P_2) = \sup \sum_i |P_1(A_i) - P_2(A_i)|,$$

where sup is taken over all finite partitions of the space Y. Thus we can talk about the metric space of the distributions P. Assume that the distribution $P(dy|x, \theta)$ is dominated by the measure $m(dy)$. The distribution density $P(\cdot|x, \theta)$ with respect to $m(dy)$ is denoted by $P(y|x, \theta)$. Assume also that θ is given by σ-algebra F, i.e. (Θ, F) is a measurable space. Define a problem of statistical identification of the distribution law $P(dy|x, \theta)$, that is, it is necessary to construct such a sequence of estimates $P_t(dy|x^t)$, that if $t \to \infty$ then

$$P_t(dy|x^t) \to P(dy|x_t, \theta)$$

in some fixed sense (in probability, with probability 1, etc).

The main apparatus in establishing of convergence is an information measure of a mismatch between the estimated distribution and evaluation. Let $P_1(y)$ and $P_2(y)$ be two distribution densities on a measurable space Y with respect to a positive measure $m(dy)$.

Definition 4. *The value*

$$I(P_1(y), P_2(y)) = \int\limits_Y \left[\ln \frac{P_1(y)}{P_2(y)}\right] P_1(y) m(dy) \tag{4}$$

is called the Kullback-Leibler information number.

The value of I is not a metric (since it is not a symmetric function of P_1 and P_2, but it has some properties that can be used as a measure of mismatch between the distribution of $P_2(y)$ and its estimate of $P_1(y)$. If integral Eq. (4) exists, then $I(P_1(g), P_2(g)) \geq 0$. Herewith $I(P_1(g), P_2(g)) = 0$ if and only if $P_1 = P_2$ is almost everywhere. The following inequality [4]

$$I(P_1(y), P_2(y)) \geq \frac{1}{8}\left[\int\limits_Y |P_1(y) - P_2(y)| m(dy)\right]^2$$

holds. Thus, it is possible to estimate the metric ρ from above by value $\sqrt{8I}$ i.e. $I \to 0$ is enough to show that $\rho \to 0$.

For the family of normal distributions another property holds. Let $P_0(x)$ be an arbitrary distribution in R^n centered at a with a covariance matrix V. Consider a family of normal distributions Q in R^n, $q_0(x)$ is the normal distribution with the same center at a and the covariance matrix V.

Theorem 1. *For every $q(x) \in Q$ the following inequality*

$$I(P_0, q_0) \leq I(P_0, q)$$

holds. The equality is achieved if and only if $q \equiv q_0$.

3 The Bayesian Approach for Solving the Problem

Consider the Bayesian approach to estimating the parameters of the conditional distribution. Let the process x_t be observed. It is assumed that the conditional distribution $P(dx_{t+1}|x_t, \theta)$ depends on a parameter $\theta \in \Theta$, for which its value is unknown but is constant throughout the experiment. Furthermore we assume that

$$P_\theta(x_{t+1}|x^t) = P(x_{t+1}|x^t, \theta).$$

The problem of this section is to study the Bayesian estimation for the transition function $P_\theta(x_{t+1}|x^t)$. A process x_t defined by $P_\theta(x_{t+1}|x^t)$ can be interpreted as a partially observed, controlled Markov process in phase space $\Theta \times x$.

Let us use the well-known construction for building an auxiliary model of the process with full information. We consider a space $\{X, N\}$ as a phase space where N is the space of distributions on Θ. Define a transition function in the space $\{X, N\}$

$$P(x_{t+1}|x_t, \nu_t) = \int_{\Theta} P_\theta(x_{t+1}|x_t)\nu_t(\theta)n(d\theta). \tag{5}$$

Definition 5. *A function $\nu_t(\theta)$ is called a posteriori distribution density of the parameter θ if it is defined by the formula*

$$\nu_{t+1}(\theta) = \nu_t(\theta)\frac{P_\theta(x_{t+1}|x_t)}{P(x_{t+1}|x_t, \nu_t)}.$$

The initial distribution and a priori distribution density $\nu_1(\theta) = d\nu_1/dn$ are also defined. Note some properties of a sequence $\nu_t(\theta)$. Let $v(\theta)$ be an arbitrary function for which there is an integral

$$\xi_t = \int_{\Theta} v(\theta)\nu_t(\theta)n(d\theta)$$

and at the same time the ratio

$$E(\xi_{t+1}|x_t, \nu_t) = \xi_t, \tag{6}$$

is true, that is, the sequence ξ_t is a martingale (with respect to $\{x_t, \nu_t\}$). Since

$$\xi_{t+1} = \int_{\Theta} v(\theta)\nu_t(\theta)\frac{P_\theta(x_{t+1}|x_t)}{P(x_{t+1}|x_t, \nu_t)}n(d\theta),$$

then

$$\int_X \xi_{t+1}P(x_{t+1}|x_t, \nu_t)m(dx_{t+1}) = \int_X m(dx_{t+1}) \int_{\Theta} v(\theta)\nu_t(\theta)P_\theta(x_{t+1}|x_t)n(d\theta) =$$

$$= \int_{\Theta} n(d\theta)v(\theta)\nu_t(\theta) \int_X P_\theta(x_{t+1}|x_t)m(dx_{t+1}) = \int_{\Theta} v(\theta)\nu_t(\theta)n(d\theta).$$

Let

$$\sigma_t^2 = \int_{\Theta} |v(\theta) - \xi_t|^2\nu_t(\theta)n(d\theta).$$

The next equality

$$E(\sigma_{t+1}^2|x_t, \nu_t) = \sigma_t^2 - E\{(\xi_{t+1} - \xi_t)^2|x_t, \nu_t\}$$

holds. Thus, the sequence σ_t^2 is a supermartingale. Let Θ be a linear set in R^n, $\bar{\theta}_t$ be the mean vectors θ with a distribution density $\nu_t(\theta)$, S_t be a correlation matrix. Then

$$E(\bar{\theta}_{t+1}|x_t, \nu_t) = \bar{\theta}_t; \quad E\{S_{t+1}|x_t, \nu_t\} \leq S_t.$$

To verify the last inequality, let $\varphi(\theta) = c^T\theta$, where c^T is a constant vector. Then $\sigma_t^2 = c^T S_t c$ and for any vector c the inequality

$$E\left\{c^T S_{t+1} c | x_t, \nu_t\right\} \leq c^T S_t c$$

holds. From Eq. (6) we have

$$E(\xi_{t+1} - \zeta_t)^2 = E\sigma_t^2 - E\sigma_{t+1}^2.$$

As a consequence of the above calculations the inequality

$$E\sum(\xi_{T+1} - \xi_T)^2 < \Sigma_1^2$$

follows. In particular, for $\xi_t = C^T\theta$ we have

$$E\left\{\sum_{T=1}^{\infty} c^T \|\bar{\theta}_{T+1} - \bar{\theta}_T\|^2\right\} < \Sigma_1^2.$$

Since the vector c is arbitrary, then

$$E\left\{\sum_{T=1}^{\infty} \|\bar{\theta}_{T+1} - \bar{\theta}_T\|^2\right\} < \Sigma_1^2.$$

Consider asymptotic properties of the sequence ν_t. Define a transition process function $\{x_t, \lambda_t\}$ by relations

$$P(x_{t+1}|x_t, \lambda_t) = \int_{\Theta} P_\theta(x_{t+1}|x_t)\lambda_t(\theta)n(d\theta), \tag{7}$$

$$\lambda_{t+1}(\theta) = \lambda_t(\theta)\frac{P_\theta(x_{t+1}|x_t)}{P(x_{t+1}|x_t, \lambda_t)}. \tag{8}$$

It is assumed that x_t is observed but the distribution $\lambda_t(\theta)$ is unknown. As an evaluation function $\lambda_t(\theta)$, we select the distribution $\nu_t(\theta)$ determined by the next recurrence relation

$$\nu_{t+1}(\theta) = \nu_t(\theta)\frac{P_\theta(x_{t+1}|x_t)}{P(x_{t+1}|x_t, \nu_t)}$$

with an arbitrary initial value of $\nu_1(\theta)$. Our task is to establish that for $T \to \infty$ the transition function $P(x_{t+1}|x_t, \nu_t)$ becomes close to "true" $P(x_{t+1}|x_t, \lambda_t)$.

Theorem 2. *Let $\{x_t, \lambda_t\}$ be the Markov process defined by Eqs. (7) and (8). Then the following inequality*

$$E\left(\sum_{t=1}^{\infty} I_t\right) \leq \int_{\Theta} \ln\left[\frac{\lambda_1(\theta)}{\nu_1(\theta)}\right]\lambda_1(\theta)n(d\theta) \tag{9}$$

holds where $I_t = I(P(\cdot|x_t), \lambda_t), P(\cdot|x_t, \nu_t))$.

Corollary 1. *The next statements*

$$\sum_{t=1}^{\infty} I_t \le +\infty;\ \lim_{t\to\infty} I_t = 0$$

are true with probability 1.

A distribution $\lambda_1(\theta)$ is arbitrary. For example, as $\lambda_1(\theta)$ we can take any distribution concentrated in an arbitrarily small neighbourhood u_0 of the point θ_0. Then all distributions $\lambda_1(\theta)$ are concentrated in this neighbourhood.

The process $\{x_t, \lambda_t\}$ can be interpreted as follows: θ_t is a random variable with a distribution $\lambda_1(\theta)$, x_{t+1} is a random variable with a conditional distribution $P_\theta(x_{t+1}|x_t)$ and $\theta_t \in u_0$ for all $t \ge 1$. Thus, the process x_t can be interpreted as the process for which a transient function is determined by a parameter subject to "small" perturbations. Thus, the convergence of the estimation of the transition function to a nonstationary transition function $P_\theta(x_{t+1}|x_t)$ is established. If the set Θ is finite, then the neighborhood of the point θ_I coincides with the point θ_I (you can choose the discrete topology). In this case, integral relations take the form of finite or infinite sums and the functions $\nu_T(\theta)$ turn into probabilistic sequences $\nu_t(i), i = 1, 2, \ldots$. Ratio Eq. (9) takes the form

$$E\left(\sum_{t=1}^{\infty} I_t\right) \le -\ln \nu_1(i_0),$$

where i_0 is the number of the transition function defined by the process x_t;

$$I_t = I(P_{i_0}; P_t),\ P_t = \sum_{t=1}^{\infty} P_i(x_{t+1}|x_t)\nu_t(i).$$

The question is whether the distribution ν_t converging to the distribution δ focused on the true parameter θ_0 remains generally open. It is easy to give an example when it is not true.

In the described approach it is assumed that the true distribution $P_0(y|x^t)$ belongs to the class of hypotheses P.

Obviously, the wider the class P is, the more computationally expensive is. Assume that the following conditions take place:

1. $\exists v_0 \in \Theta : v_0 = \left\{\Theta \,|\, E\left\{\ln \frac{P_0}{P_\theta}\right\} < +\infty\right\}$;
2. $a = \inf\limits_{\theta\in v_\epsilon} E\left\{\ln \frac{P_0}{P_\theta}\right\};\ \delta > 0;\ \exists v_\delta \subset v_0;\ E\left\{\sup\limits_{\theta\in v_\delta} I(P_0; P_\theta) < a + \delta\right\},$
   $\theta \in v_\delta,\ \left|\ln \frac{P_0}{P_\theta}\right|^2 < c < +\infty,\ \int\limits_{v_\delta} \nu_0(\theta)\, n(d\theta) > 0;$
3. $\ln P_\theta$ has finite the mathematical expectation and the dispersion.

Denote

$$l(\theta, x_t) = \int\limits_Y [\ln P_\theta(y|x_t)]\, P_0(y|x_t) m(dy).$$

Example 1. Give an example illustrating the introduced conditions. Let

$$y_t = \theta^T x_t + f_t,$$

where x_t, y_t are observed vectors; x_t is an independent random vector; f_t is an unobservable sequence of random variables. The "true" hypothesis $P(y|x_t, \theta)$ has the form

$$P(y|x_t, \theta) = \varphi\left(y_t - \theta^T x_t\right),$$

where φ is the density f. If the density f is unknown and if the family of normal distributions is taken as hypotheses, then condition 2 means that vectors x_t have finite fourth moments.

Theorem 3. *If $\varphi(\theta_0) > \varphi(\theta_1)$, then for any N the relation*

$$P\left\{\frac{\nu_t(\theta_0)}{\nu_t(\theta_1)} > N\right\} \to 1, \ \text{if} \ \ t \to \infty,$$

holds

Theorem 4. *If $\theta \notin v_\delta$ then $\nu_t(\theta) \to 0$ in probability if $t \to \infty$.*

Now back to Eq. (3). The Bayesian construction defined by formulas Eqs. (5), (6) and associated to the family of transition functions $P_\theta(\cdot|z_t, w_t)$ has the form

$$P(x_{t+1}|\nu_t, z_t, w_t) = \int\limits_{\Theta} F(x_{t+1} - \theta^T z_t - w_t)\nu_t(\theta)n(d\theta),$$

$$\nu_{t+1}(\theta) = \nu_t(\theta)\frac{F(x_{t+1} - \theta^T z_t - w_t)}{P(x_{t+1}|\nu_t, z_t, w_t)}.$$

Let $\nu_t(\theta)$ be the density of a normal random vector with an average value of θ_t and a correlation matrix S_t:

$$\nu_t(\theta) = \Gamma_t \exp\left\{-\frac{1}{2}(\bar{\theta}_t - \theta_t)^T S_t^{-1}(\bar{\theta}_t - \theta_t)\right\}.$$

Then $\nu_{t+1}(\theta)$ is the density of the normal vector with the correlation matrix S_{t+1} and the average value of θ_{t+1} while the following recurrence relations analogous to the relations of the discrete Kalman filter

$$S_{t+1}^{-1} = S_t^{-1} + \frac{1}{\sigma_f^2} z_t z_t^T,$$

$$\theta_{t+1} = S_{t+1}[S_t^{-1}\theta_t + \frac{1}{\sigma_f^2}(x_{t+1} - w_t)z_1]. \tag{10}$$

are valid. From relation Eq. (10) it follows that in the case of the Gaussian quantities f_t a posteriori distribution densities $\nu_t(\theta)$ have an interesting property. Namely their correlation matrices are monotonically decreasing $S_{t+1} \leq S_t$.

Denote by $z_t^T = (x_t, x_{t-1}, ..., x_{t-n})$ the phase vector of equation Eq. (2), then this equation can be rewritten as follows

$$z_{t+1} = Az_t + b(\theta^T z_t + \sum_{k=0}^{m} \beta_k u_{t-k} + f_{t+1}), \tag{11}$$

where

$$A = \begin{pmatrix} 00...00 \\ \\ 10...00 \\ \cdots \\ 00...01 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $V(z)$ be an arbitrary quadratic form of a vector z. Calculate the conditional expectation $E\{V(z_{t+1})|z_t, \nu_t, u_t\}$. As the consequence of the above, to calculate the conditional distribution of the vector z_{t+1} in the Bayesian construction defined by relations Eqs. (5) and (6), in formula Eq. (11) we should consider θ as a random vector with the distribution $\nu_t(\theta)$. Therefore

$$E\{V(z_{t+1})|z_t\} = V(b)Ef_{t+1}^2 + \int_{\Theta} V\left(Az_t + b\left(\theta^T z_t + \sum_{k=0}^{m} \beta^k u\right)\right)\nu_t(\theta)d\theta =$$

$$= V(b)Ef_{t+1}^2 + V\left(Az_t + b\left(\theta^T z_t + \sum_{k=0}^{m} \beta^k u_{t-k}\right)\right) + V(b)\int_{\Theta} |(\theta - \theta_t)^T z_t|^2 \nu_t(\theta)d\theta,$$

where

$$\theta_t = \int_{\Theta} \theta\nu_t(\theta)d\theta. \tag{12}$$

It is obvious that

$$\int_{\Theta} |(\theta - \theta_t)^T z_t|^2 \nu_t(\theta)d\theta = z_t^T S_t z_t,$$

where

$$S_t = \int_{\Theta} (\theta - \theta_t) * (\theta - \theta_t)^T \nu_t(\theta)d\theta.$$

Finally write

$$E\{V(z_{t+1})|x^t\} = qd^2 + qz_t^T S_t z_t + V\left(Az_t + b\left(\theta_t^T z_t + \sum_{k=0}^{m} \beta^k u_{t-k}\right)\right). \tag{13}$$

Let the quadratic form $V(z)$ have the form

$$V(z_t) = nx_t^2 + (n-1)x_{t-1}^2 + ... + x_{t-n+1}^2.$$

Then

$$V(Az_t + bv) + z_t^2 = V(z) + nv^2;\ V(b) = n.$$

Relation Eq. (13) can be rewritten as

$$E(V(z_{t+1})|x^t) + (z_t^2 - qz_t^T S_t z_t) = n\left(d^2 + \left|\theta_t^T z_t + \sum_{k=0}^{m} \beta^k u_{t-k}\right.\right) + V(z_t).$$

Theorem 5. *The strategy* $\sum\limits_{k=0}^{m} \beta_k u_{t-k} = -\theta_t^T z_t$ *minimizes the value*

$$J_T = E_\nu \left\{ \sum_{t=0}^{T-1} z_t^T (I - S_T n) z_t + V(z_T) \right\}$$

at each T. *Herewith*

$$\inf_u J_T = nTd^2 + V(z_0).$$

Corollary 2. *Let for some* t *the inequality* $I - qS_T > \epsilon_0 I$ *be satisfied. Then there exists* $\rho,\ 0 < \rho < 1$, *for which the inequality*

$$E\{V(z_{t+1})|z_t, \nu_t\} \le \rho V(z_t) + nd^2$$

holds.

Corollary 3. *Let* $\lambda(\theta)$ *be a priori density for which the inequality* $\lambda(\theta) \le C\gamma(\theta)$ *holds. Then the following inequality*

$$E_\lambda \left(\sum_t^T z_t^2 \right) \le C E_\nu \left(\sum_t^T z_t^2 \right)$$

holds.

Theorem 6. *Let*

$$\sum_{k=0}^{m} \beta_k u_{t-k} = -\theta_t^T z_t,$$

where θ_t *is defined by Eq. (12),* $\nu_t(\theta)$ *be a sequence of a posteriori distributions corresponding to the normal a priori distribution* $\nu_t(\theta)$ *and* $nS_1 < I$. *Let* $\lambda(\theta)$ *be the density of distribution concentrated at a bounded domain and satisfying the inequality* $\lambda(\theta) < C\gamma(\theta)$. *Then*

$$E\{z_t^2\} \le const, \quad E\{u_t^2\} \le const.$$

Remark 1. Let us explain the meaning of the statements of this theorem. The process $(x_t; \nu_t)$ defined by the transition function $P(x_{t+1}|z_t, \nu_t, u_t)$ can be interpreted as the process defined by the equation

$$x_{t+1} = \theta_t^T z_t + \sum_{k=0}^{m} \beta^k u_{t-k} + f_{t+1},$$

where θ_t is a random vector with a distribution $\lambda_t(\theta)$. Since the carrier distributions of $\lambda_t(\theta)$ are centered in the neighbourhood of the point θ_0, then for all t the inequality $|\theta_t - \theta_0| < \epsilon$ holds.

Thus, the theorem states that the strategy stabilizes the trajectory of equation Eq. (11) with a "small" random perturbation of the vector coefficients of equation Eq. (11). Also we can definitely hope that this strategy stabilizes the trajectory and for the equation with the unperturbed parameter vector $\theta_t \equiv \theta_0$.

4 Conclusion

The problem of convergence and stability of the Bayesian parameter estimation of the linear stochastic control system is considered. The problem of synthesis of optimal strategies for dynamic systems in which there is no part of the basic information necessary for the construction of optimal control is formulated. The optimal strategy which stabilizes the trajectory of a linear stochastic control system with a "small" random perturbation of the coefficient vector is obtained. The Bayesian parameter estimation is widely used in many areas such as medicine, biology and economics and many others [2,5–9].

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