

Periodic INAR(1) Models with Skellam-Distributed Innovations

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Abstract. In this paper, an integer-valued autoregressive model of order one (INAR(1)) with time-varying parameters and driven by a periodic sequence of innovations is introduced. The proposed INAR(1) model is based on the signed thinning operator defined by Kachour and Truquet (2011) and conveniently adapted to the periodic case. Basic notations and definitions concerning the periodic signed thinning operator are provided. Based on this thinning operator, Chesneau and Kachour (2012) established a signed INAR(1) model. Motivated by the work of Chesneau and Kachour (2012), we introduce a periodic model, denoted by S-PINAR(1), with period s. In contrast to conventional INAR(1) models, these models are defined in $\mathbb{Z}$ allowing for negative values both for the series and its autocorrelation function. For a proper $\mathbb{Z}$-valued time series, a distribution for the innovation term defined on $\mathbb{Z}$ is required. The S-PINAR(1) model assumes a specific innovation distribution, the Skellam distribution. Regarding parameter estimation, two methods are considered: conditional least squares and conditional maximum likelihood. The performance of the S-PINAR(1) model is assessed through a simulation study.

Keywords: Integer-valued autoregressive models · Signed thinning operator · Skellam distribution

1 Introduction

The class of INAR models, based on the binomial thinning operator introduced by Steutel and van Harn (1979), can only be applied to count variables, i.e., to non-negative integer-valued r.v.'s as their range, therefore, cannot account for negative integers. Whilst models for non-negative integer-valued time series are now abundant, there is a shortage of similar models when the time series refer to

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data defined on $\mathbb{Z}$, i.e., in both the positive and negative integers. The need for such models can also appear when taking differences of positive integer-valued count time series.

The binomial thinning operator has been generalized in a number of different ways. The first model for data with range in $\mathbb{Z}$ was introduced by Kim and Park (2008). Their model was based on the signed binomial thinning operator, allowing time series with negative values. A particular case of this model can be found in Andersson and Karlis (2014).

Using a slightly different version of the signed thinning operator defined by Kim and Park (2008), Kachour and Truquet (2011) focused on a more general class of $\mathbb{Z}$-valued processes denoted by SINAR (Signed INAR). This modified version of the thinning operator, called the signed thinning operator, allows for negative values both for the series and its autocorrelation function. Kachour and Truquet (2011) point out that the signed thinning operator is the natural extension of the Steutel and van Harn operator to $\mathbb{Z}$-valued random variables. The authors avoid, however, a parametric assumption for the innovation term. Based on the signed thinning operator and under a parametric assumption on the common distribution of the counting sequence of the model, Chesneau and Kachour (2012) focus on the simple SINAR(1) model.

For an adequate time series on $\mathbb{Z}$ we also need to consider a distribution for the innovation term defined on $\mathbb{Z}$. The literature is limited on this subject. However, discrete distributions defined on the set of integers has attracted the attention of several researchers. Two ways to define distributions on $\mathbb{Z}$ are: the differences between two non-negative discrete r.v.'s and the discrete version of continuous distributions on $\mathbb{R}$. The main distributions on the set $\mathbb{Z}$ are Poisson difference, discrete normal and discrete Laplace. The Poisson difference distribution, also known as the Skellam distribution, is traditionally linked to Skellam (1946) and has found applications in areas such as sports (Karlis and Ntzoufras 2009) and finance (Alzaid and Omair 2010). The extended binomial distribution was introduced as an alternative to the Skellam distribution by Alzaid and Omair (2012). A natural $\mathbb{Z}$-extension of the INAR model, originally defined on $\mathbb{N}$, was presented by Alzaid and Omair (2014). We also mention the extended Poisson distribution introduced by Bakouch *et al.* (2016), the first version of the Poisson distribution over the set of all integers.

In this paper, a new first-order integer-valued autoregressive model with time-varying parameters and sequences of innovations with periodic structure is established. Motivated by the work of Chesneau and Kachour (2012), we extended their model based on the signed thinning operator to the periodic case. A signed periodic INAR(1) process (S-PINAR(1) for short) with period s is introduced. In contrast to traditional INAR(1) models, these models are defined in $\mathbb{Z}$ allowing for negative integer values and negative correlation. The properties of the S-PINAR(1) model are discussed. Focus is placed upon a specific parametric case which arises under the assumption of periodic Skellam-distributed innovations. Two methods are considered for parameter estimation: conditional least squares and conditional maximum likelihood. A modification of the traditional condi-

tional least squares method was made through a two step procedure in order to provide estimators for all parameters involved in the periodic univariate model. The performance of the proposed estimation methods for the S-PINAR(1) model with period s is accomplished and compared through a simulation study, contemplating six different combinations of the parameters.

In this section, basic notations and definitions concerning the periodic signed thinning operator are presented. A brief description of the periodic Skellam distribution defined on the whole set of integers is also provided.

1.1 Periodic Signed Thinning Operator

The definition of the signed thinning operator introduced by Kachour and Truquet (2011), adapted to the periodic case, is defined by

$$F_t \odot X = \begin{cases} sign(X) \sum_{i=1}^{|X|} U_{i,t}(\phi_t) \ , X \neq 0 \\ 0 \qquad\qquad\quad , \ otherwise \end{cases} \tag{1}$$

with $sign(X) = 1, x \geq 0$ and -1 otherwise and where F_t represents the common distribution of the periodic sequence of i.i.d. counting sequences $(U_{i,t}(\phi_t))_{i\in\mathbb{N}}$. All counting sequences associated to the operator $F_t\odot$ are mutually independent.

We consider that F_t, the distribution of the periodic sequence of i.i.d. random variables $(U_{i,t}(\phi_t))_{i\in\mathbb{N}}$, has probability mass function given by

$$P(U_{1,t}(\phi_t) = a) = \begin{cases} (1-\phi_t)^2, & a = -1 \\ 2\phi_t(1-\phi_t), & a = 0 \\ \phi_t^2, & a = 1 \end{cases},$$

with $\phi_t = \alpha_v \in (0,1)$ for $t = v + ns; v = 1, \ldots, s$ and $n \in \mathbb{N}_0$. Chesneau and Kachour (2012) have also made use of this common distribution but without the periodic structure. Note that, for a fixed v, the random variable

$$U_{i,t}(\phi_t) = U_t(\phi_t) \stackrel{d}{=} R_t(\phi_t) - 1, \qquad R_t(\phi_t) \sim Bin(2, \phi_t) \tag{2}$$

and

$$P\left(\sum_{i=1}^{k} U_{i,t}(\phi_t) = l\right) = P\left(R_t^{(k)}(\phi_t) = k + l\right), \qquad l \in \{-k, \ldots, k\},$$

where

$$R_t^{(k)}(\phi_t) = \sum_{i=1}^{k} R_{i,t}(\phi_t), \qquad R_t^{(k)}(\phi_t) \sim Bin(2k, \phi_t), \ k \in \mathbb{N}. \tag{3}$$

Then, for $x \in \mathbb{Z}\backslash\{0\}$ and $y \in \mathbb{Z}$, the conditional probability function of the periodic signed thinning operator $F_t\odot$ defined in (1) is

$$P\left(F_t \odot X = y | X = x\right) = P\left(sign(x)\sum_{i=1}^{|x|} U_{i,t}(\phi_t) = y\right)$$

$$= P\left(R_t^{|x|}(\phi_t) - |x| = sign(x)\cdot y\right) = P\left(R_t^{|x|}(\phi_t) = |x| + sign(x)\cdot y\right)$$

$$= C_{|x|+sign(x)\cdot y}^{2|x|}\alpha_v^{|x|+sign(x)\cdot y}(1-\alpha_v)^{|x|-sign(x)\cdot y}, \qquad y \in \{-|x|,\ldots,|x|\}$$

with mean value

$$E[F_t \odot X|X] = (2\alpha_v - 1)X \tag{4}$$

and variance

$$Var[F_t \odot X|X] = 2\alpha_v(1-\alpha_v)|X| \tag{5}$$

for $t = v + ns$, $v = 1,\ldots,s$ and $n \in \mathbb{N}_0$.

1.2 Periodic Skellam Distribution

Let $\{Z_t\}, t = v + ns;\ v = 1,\ldots,s$ and $n \in \mathbb{N}_0$ be a periodic sequence of random variables. For a fixed v $(v = 1,\ldots,s)$, let $\lambda_v > 0$ and $\tau_v > 0$. The periodic s-dimensional r.v. Z_t follows a periodic Skellam distribution, denoted by $Skellam(\lambda_v, \tau_v)$, if and only if

$$Z_{v+ns} \stackrel{d}{=} Y_{v+ns} - W_{v+ns},$$

where Y_{v+ns} and W_{v+ns} are two independent r.v.'s such that $Y_{v+ns} \sim Poisson(\lambda_v)$ and $W_{v+ns} \sim Poisson(\tau_v)$.

Thus, the probability mass function is given by

$$P(Z_{v+ns} = z) = e^{-(\lambda_v+\tau_v)}\lambda_v^z \sum_{i=max(0,-z)}^{\infty} \frac{(\lambda_v\tau_v)^i}{i!(i+z)!},\ z \in \mathbb{Z}. \tag{6}$$

The random vector Z_t has finite first and second-order moments. The mean of Z_t, $t = v + ns$ for a fixed v $(v = 1,\ldots,s)$, is

$$\xi_v = E[Z_{v+ns}] = E[Y_{v+ns} - W_{v+ns}] = \lambda_v - \tau_v. \tag{7}$$

Due to the independence of the r.v.'s Y_{v+ns} and W_{v+ns}, the variance of Z_t for $t = v + ns$ with a fixed v is

$$\sigma_v^2 = Var[Z_{v+ns}] = Var[Y_{v+ns} - W_{v+ns}] = \lambda_v + \tau_v. \tag{8}$$

The probability generating function of Z_{v+ns} is

$$G_{Z_{v+ns}}(r) = exp\{-(\lambda_v + \tau_v) + \lambda_v r + \tau_v/r\},\ v = 1,\ldots,s.$$

2 The Univariate Periodic Model: S-PINAR(1)

Let $\{X_t\}$ be a periodic integer-valued autoregressive process of first-order defined by the recursion

$$X_t = F_t \odot X_{t-1} + Z_t, \ t \in \mathbb{Z}, \tag{9}$$

where X_t, X_{t-1} and Z_t are random s-vectors for $t = v+ns$ with $v = 1, \ldots, s$ and $n \in \mathbb{N}_0$. The random vector $Z_t = [Z_{1+ns} \ Z_{2+ns} \ldots \ Z_{s+ns}]^T$ represents a periodic sequence of independent random variables. The model defined in Eq. (9) will be referred to as `S-PINAR`(1) for `S`igned `P`eriodic `IN`teger-valued `A`uto`R`egressive model of order one with period $s \in \mathbb{N}$. For each t, the innovation term Z_t in recursion (9) is assumed to be independent of X_{t-1} and $F_t \odot X_{t-1}$. Writing the periodic signed thinning operator in (1) as

$$F_t\odot = \begin{cases} f_1\odot, t = 1+ns \\ f_2\odot, t = 2+ns \\ \quad \vdots \\ f_s\odot, t = s+ns \end{cases},$$

the periodic model in (9) can have the form $X_{v+ns} = f_v \odot X_{v-1+ns} + Z_{v+ns}$ where $f_v \odot X_{v-1+ns} = sign(X_{v-1+ns}) \sum_{i=1}^{|X_{v-1+ns}|} U_{i,t}(\phi_t)$ with $U_{i,t}(\phi_t)$ as in (2).

We assume the innovation term Z_t in the `S-PINAR`(1) model proposed in (9) follows the periodic Skellam distribution with parameters λ_v and τ_v established in Subsect. 1.2 with probability mass function given by Eq. (6). Therefore, for a fixed v with $v = 1, \ldots, s$, the first and second-order moments of Z_{v+ns} are defined in (7) and (8), respectively.

Some distributional properties of the `S-PINAR`(1) process in recursion (9) with Skellam-distributed innovation are derived, namely the conditional moments of first and second-order of the model. Hence, from (4) and (7)

$$\begin{aligned} E[X_{v+ns}|X_{v-1+ns}] &= E[f_v \odot X_{v-1+ns} + Z_{v+ns}|X_{v-1+ns}] \\ &= (2\alpha_v - 1)X_{v-1+ns} + \lambda_v - \tau_v \end{aligned} \tag{10}$$

and from Eqs. (5) and (8),

$$\begin{aligned} Var[X_{v+ns}|X_{v-1+ns}] &= Var[f_v \odot X_{v-1+ns} + Z_{v+ns}|X_{v-1+ns}] \\ &= 2\alpha_v(1-\alpha_v)|X_{v-1+ns}| + \lambda_v + \tau_v. \end{aligned} \tag{11}$$

For a fixed value of $v = 1, \ldots, s$, the process $\{X_t\}$ with $t = v+ns$ is a Markov chain with transition probability function

$$\begin{aligned} p_v(b|a) &= P(X_{v+ns} = b|X_{v-1+ns} = a) \\ &= \sum_{l=-|a|}^{|a|} P\left(sign(a)\sum_{i=1}^{|a|} U_{i,t}(\phi_t) = l\right) \times P(Z_{v+ns} = b-l) \end{aligned}$$

$$= \sum_{l=-|a|}^{|a|} P\left(R_t^{(|a|)}(\phi_t) = |a| + sign(a)\cdot l\right) \times P(Z_{v+ns} = b - l)$$

$$= \sum_{l=-|a|}^{|a|} \left\{ C_{|a|+sign(a)\cdot l}^{2|a|} \alpha_v^{|a|+sign(a)\cdot l} (1-\alpha_v)^{|a|-sign(a)\cdot l} \right.$$

$$\left. \times\ e^{-(\lambda_v+\tau_v)} \lambda_v^{b-l} \sum_{i=max(0,-(b-l))}^{\infty} \frac{(\lambda_v\tau_v)^i}{i!(i+b-l)!} \right\}, \tag{12}$$

where the probability mass function of $R_t^{(|a|)}$ and Z_{v+ns} can be found in (3) and (6), respectively.

3 Parameter Estimation

This section is devoted to parameter estimation of the `S-PINAR`(1) process with period s under the parametric assumption previously mentioned. Lets us assume we have $(X_0, X_1, \ldots, X_{Ns})$ observations from the `S-PINAR`(1) process with Skellam-distributed innovations. Two estimation methods are proposed to estimate the parameters of the model: conditional least squares and conditional maximum likelihood. For the `S-PINAR`(1) model with period s, the vector of unknown parameters $\boldsymbol{\theta}$ has $3s$ parameters, i.e.,

$$\boldsymbol{\theta} := (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\tau}) \tag{13}$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_s)$.

3.1 Conditional Least Squares Estimation

The conditional least squares (CLS) estimator of the vector of the unknown parameters in (13) is $\widehat{\boldsymbol{\theta}}_{CLS} := (\widehat{\boldsymbol{\alpha}}^{CLS}, \widehat{\boldsymbol{\lambda}}^{CLS}, \widehat{\boldsymbol{\tau}}^{CLS})$. The estimation procedure that follows was proposed by Klimko and Nelson (1978). The CLS estimators of $\boldsymbol{\theta}$ are obtained by minimizing the criterion function $S_1(\boldsymbol{\theta})$ given by

$$S_1(\boldsymbol{\theta}) = \sum_{n=0}^{N-1} \sum_{v=1}^{s} \left(X_{v+ns} - (2\alpha_v - 1)X_{v-1+ns} - \lambda_v + \tau_v\right)^2.$$

It is clear that differentiating $S_1(\boldsymbol{\theta})$ with respect to λ_v and τ_v and equating the resulting expressions to zero, the same equation is obtained. For these parameters, direct CLS estimators are not available. The conditional least squares method was adapted by Alzaid and Omair (2014) with some modifications in order to be able to estimate all parameters integrating the model. Hence, in order

to estimate λ_v and τ_v using the CLS method, the following reparametrization is needed

$$\begin{cases} \xi_v = \lambda_v - \tau_v \\ \sigma_v^2 = \lambda_v + \tau_v \end{cases}, v = 1, \ldots, s. \tag{14}$$

Estimators for all parameters of the S-PINAR(1) process, i.e., α_v, ξ_v and σ_v^2 are obtained in a two step procedure as described below.

First Step - Estimates for α_v and ξ_v $(v = 1, \ldots, s)$. Consider the conditional mean prediction error

$$\begin{aligned} e_{1,v+ns} &= X_{v+ns} - E[X_{v+ns}|X_{v-1+ns}] \\ &= X_{v+ns} - (2\alpha_v - 1)X_{v-1+ns} - \xi_v, \end{aligned} \tag{15}$$

where conditional first-order moment $E[X_{v+ns}|X_{v-1+ns}]$ is defined in (10). The CLS estimators of α_v and ξ_v are derived by minimizing the criterion function

$$S_2(\boldsymbol{\theta}) = \sum_{n=0}^{N-1} \sum_{v=1}^{s} e_{1,v+ns}^2 = \sum_{n=0}^{N-1} \sum_{v=1}^{s} \left(X_{v+ns} - (2\alpha_v - 1)X_{v-1+ns} - \xi_v\right)^2.$$

After differentiating $S_2(\boldsymbol{\theta})$ with respect to parameters α_v and ξ_v, the following system of equations arises

$$\begin{cases} \dfrac{\partial S_2(\boldsymbol{\theta})}{\partial \alpha_v} = \displaystyle\sum_{n=0}^{N-1} \left(X_{v+ns} - (2\alpha_v - 1)X_{v-1+ns} - \xi_v\right) X_{v-1+ns} = 0 \\ \dfrac{\partial S_2(\boldsymbol{\theta})}{\partial \xi_v} = \displaystyle\sum_{n=0}^{N-1} \left(X_{v+ns} - (2\alpha_v - 1)X_{v-1+ns} - \xi_v\right) = 0 \end{cases}$$

and consequently, for $v = 1, \ldots, s$, the CLS estimators are

$$\begin{cases} \widehat{\alpha}_v^{CLS} = \dfrac{1}{2}\left(\dfrac{N \displaystyle\sum_{n=0}^{N-1} X_{v+ns}X_{v-1+ns} - \sum_{n=0}^{N-1} X_{v+ns} \sum_{n=0}^{N-1} X_{v-1+ns}}{N \displaystyle\sum_{n=0}^{N-1} X_{v-1+ns}^2 - \left(\sum_{n=0}^{N-1} X_{v-1+ns}\right)^2} + 1 \right) \\ \widehat{\xi}_v^{CLS} = \dfrac{1}{N}\left(\displaystyle\sum_{n=0}^{N-1} X_{v+ns} - (2\widehat{\alpha}_v^{CLS} - 1) \sum_{n=0}^{N-1} X_{v-1+ns} \right) \end{cases}. \tag{16}$$

Second Step - Estimate for σ_v^2 $(v = 1, \ldots, s)$. The conditional variance prediction error has been used by Alzaid and Omair (2014) to obtain the CLS estimator for the variance parameter. Thus in the periodic case, the conditional variance prediction error is defined by

$$\begin{aligned} e_{2,v+ns} &= (X_{v+ns} - E[X_{v+ns}|X_{v-1+ns}])^2 - Var[X_{v+ns}|X_{v-1+ns}] \\ &= e_{1,v+ns}^2 - 2\alpha_v(1-\alpha_v)|X_{v-1+ns}| - \sigma_v^2 \end{aligned}$$

with conditional moments $E[X_{v+ns}|X_{v-1+ns}]$ and $Var[X_{v+ns}|X_{v-1+ns}]$ in Eqs. (10) and (11), respectively. The conditional mean prediction error ($e_{1,v+ns}$) is derived in the first step of the estimation procedure from (15). The equation $\sum_{n=0}^{N-1} e_{2,v+ns} = 0$ yields a direct estimator for σ_v^2 by solving the nonlinear equation

$$\sum_{n=0}^{N-1} \left(\widehat{e}_{1,v+ns}^2 - 2\widehat{\alpha}_v^{CLS}(1-\widehat{\alpha}_v^{CLS})|X_{v-1+ns}| - \sigma_v^2\right) = 0,$$

i.e.,

$$\widehat{\sigma}_v^2 = \frac{1}{N}\sum_{n=0}^{N-1} \left(\widehat{e}_{1,v+ns}^2 - 2\widehat{\alpha}_v^{CLS}(1-\widehat{\alpha}_v^{CLS})|X_{v-1+ns}|\right), \tag{17}$$

where $\widehat{e}_{1,v+ns} = X_{v+ns} - (2\widehat{\alpha}_v^{CLS} - 1)X_{v-1+ns} - \widehat{\xi}_v^{CLS}$ with CLS estimators $\widehat{\alpha}_v^{CLS}$ and $\widehat{\xi}_v^{CLS}$ in (16). After estimating σ_v^2 through (17), the CLS estimators of λ_v and τ_v from reparametrization (14) take the form

$$\begin{cases} \widehat{\lambda}_v^{CLS} = \dfrac{1}{2}\left(\widehat{\sigma}_v^{2,CLS} + \widehat{\xi}_v^{CLS}\right) \\ \widehat{\tau}_v^{CLS} = \dfrac{1}{2}\left(\widehat{\sigma}_v^{2,CLS} - \widehat{\xi}_v^{CLS}\right) \end{cases}, \quad v = 1, \ldots, s.$$

3.2 Conditional Maximum Likelihood Estimation

The conditional maximum likelihood (CML) estimator of the vector of the unknown parameters in (13) is $\widehat{\boldsymbol{\theta}}_{CML} := (\widehat{\boldsymbol{\alpha}}^{CML}, \widehat{\boldsymbol{\lambda}}^{CML}, \widehat{\boldsymbol{\tau}}^{CML})$. The conditional log-likelihood function is given by

$$C(\boldsymbol{\theta}) = ln(L(\boldsymbol{\theta}|\mathbf{x})) = \sum_{n=0}^{N-1}\sum_{v=1}^{s} ln\left(p_v(x_{v+ns}|x_{v-1+ns})\right), \tag{18}$$

where $p_v(b|a)$ has the expression given in (12) by replacing $a = x_{v-1+ns}$ and $b = x_{v+ns}$. Differentiating the conditional log-likelihood function in Eq. (18) with respect to the parameters α_v, λ_v and τ_v ($v = 1, \ldots, s$) in (13), the system of first-order partial derivatives follows

$$\begin{cases} \dfrac{\partial C(\boldsymbol{\theta})}{\partial \alpha_v} = 0 \\ \dfrac{\partial C(\boldsymbol{\theta})}{\partial \lambda_v} = 0 \\ \dfrac{\partial C(\boldsymbol{\theta})}{\partial \tau_v} = 0 \end{cases} \Leftrightarrow \begin{cases} \sum\limits_{n=0}^{N-1} \dfrac{\dfrac{\partial}{\partial \alpha_v} p_v(x_{v+ns}|x_{v-1+ns})}{p_v(x_{v+ns}|x_{v-1+ns})} = 0 \\ \sum\limits_{n=0}^{N-1} \dfrac{\dfrac{\partial}{\partial \lambda_v} p_v(x_{v+ns}|x_{v-1+ns})}{p_v(x_{v+ns}|x_{v-1+ns})} = 0 \\ \sum\limits_{n=0}^{N-1} \dfrac{\dfrac{\partial}{\partial \tau_v} p_v(x_{v+ns}|x_{v-1+ns})}{p_v(x_{v+ns}|x_{v-1+ns})} = 0 \end{cases}, \ v = 1, \ldots, s,$$

i.e.,

$$\begin{cases} \sum\limits_{n=0}^{N-1} \dfrac{2|x_{v-1+ns}|}{1-\alpha_v} \left(\dfrac{p_v(x_{v+ns}-1|x_{v-1+ns}-1)}{p_v(x_{v+ns}|x_{v-1+ns})} - 1 \right) = 0 \\ \sum\limits_{n=0}^{N-1} \dfrac{p_v(x_{v+ns}-1|x_{v-1+ns})}{p_v(x_{v+ns}|x_{v-1+ns})} = N \\ \sum\limits_{n=0}^{N-1} \dfrac{p_v(x_{v+ns}+1|x_{v-1+ns})}{p_v(x_{v+ns}|x_{v-1+ns})} = N \end{cases}, v = 1, \ldots, s.$$

Calculus for first-order partial derivatives of transition probability function $p_v(x_{v+ns}|x_{v-1+ns})$ are omitted here. Numerical maximization can be obtained through standard statistical packages in `R`.

4 Simulation Study

In order to provide an idea about the relative merits of each method (CLS and CML) used in parameter estimation of the `S-PINAR`(1) model with period s and Skellam-distributed innovation term, a simulation study is conducted. To generate count data from the periodic univariate model proposed in (9), we have set period $s = 4$, thus the vector of unknown parameters in (13) is $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\tau}) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \tau_1, \tau_2, \tau_3 \tau_4)$. Several combinations of values for parameters $\boldsymbol{\alpha}, \boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ are available in Table 1. Three sets, namely, Set 1, Set 2 and Set 3 are displayed. Each set has been subdivided into settings A and B, where parameter $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is fixed. Hence in Table 1, the different scenarios will be referred to as Set 1A, Set 1B, Set 2A, Set 2B, Set 3 A and Set 3B. For Set 1, values for α_v $(v = 1, 2, 3, 4)$ are above and below 0.5. For both settings, A and B, different values for $\boldsymbol{\lambda}$ are considered while parameter $\boldsymbol{\tau}$ remains the same. Regarding Set 2, values for α_v are all below 0.5 and both parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ take different values. For Set 3, values for α_v are all above 0.5, parameter $\boldsymbol{\lambda}$ is fixed but parameter $\boldsymbol{\tau}$ assumes different values. The choice for certain values of parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ arise from the fact that $\lambda_v - \tau_v$ represents the mean of Z_{v+ns} given in (7).

Table 1. Parameters: $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4)$.

Set 1	A:	$\boldsymbol{\alpha} = (0.60, 0.40, 0.75, 0.30)$;	$\boldsymbol{\lambda} = (2, 1, 6, 5)$;	$\boldsymbol{\tau} = (4, 5, 3, 1)$
	B:	$\boldsymbol{\alpha} = (0.60, 0.40, 0.75, 0.30)$;	$\boldsymbol{\lambda} = (5, 2, 1, 6)$;	$\boldsymbol{\tau} = (4, 5, 3, 1)$
Set 2	A:	$\boldsymbol{\alpha} = (0.20, 0.45, 0.10, 0.30)$;	$\boldsymbol{\lambda} = (2, 1, 6, 5)$;	$\boldsymbol{\tau} = (4, 5, 3, 1)$
	B:	$\boldsymbol{\alpha} = (0.20, 0.45, 0.10, 0.30)$;	$\boldsymbol{\lambda} = (5, 2, 1, 6)$;	$\boldsymbol{\tau} = (2, 1, 4, 3)$
Set 3	A:	$\boldsymbol{\alpha} = (0.75, 0.62, 0.51, 0.86)$;	$\boldsymbol{\lambda} = (4, 5, 3, 1)$;	$\boldsymbol{\tau} = (1, 3, 2, 4)$
	B:	$\boldsymbol{\alpha} = (0.75, 0.62, 0.51, 0.86)$;	$\boldsymbol{\lambda} = (4, 5, 3, 1)$;	$\boldsymbol{\tau} = (2, 1, 4, 3)$

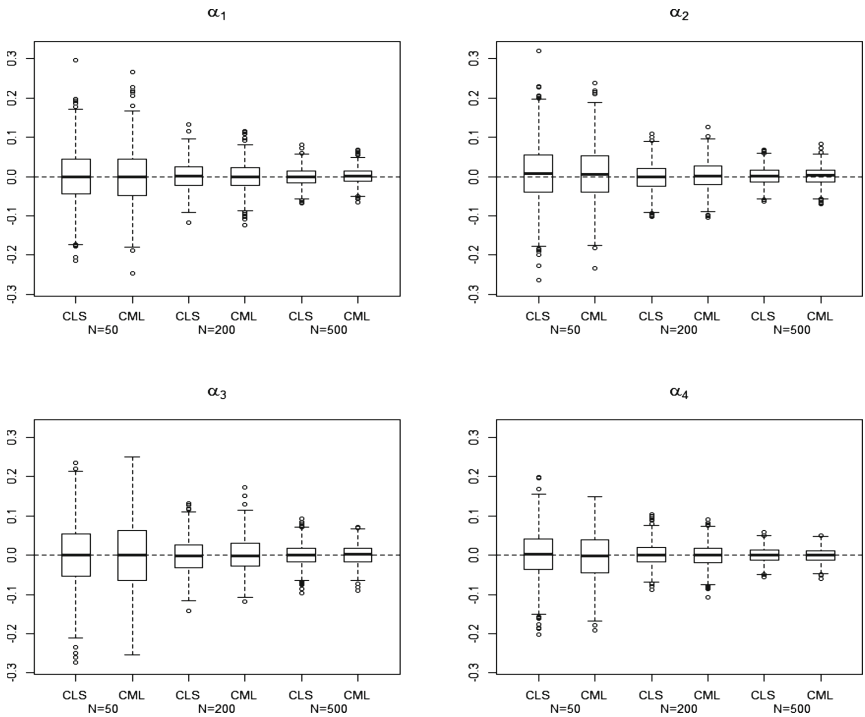


Fig. 1. Boxplots for the biases of the CLS and CML estimates of parameter $\boldsymbol{\alpha}$ in Set 1 A for $n = 4N = 200, 800, 2000$.

Three sample sizes are contemplated in this simulation study. For cycles $N = 50, 200, 500$, $n = 4N = 200, 800, 2000$. For a fixed set of parameters in Table 1, 1000 independent replications of the `S-PINAR`(1) process have been generated. All simulation and estimation procedures were realized through functions written in `R`. Table 2 reports the average parameter estimates for Set 1A. To facilitate comparison between the CLS and CML methods and the aforementioned sample sizes, the mean square error (MSE) was computed and included in parenthesis below each estimate. According to Table 2, parameter estimates in both cases are very close, because both methods give consistent estimates of

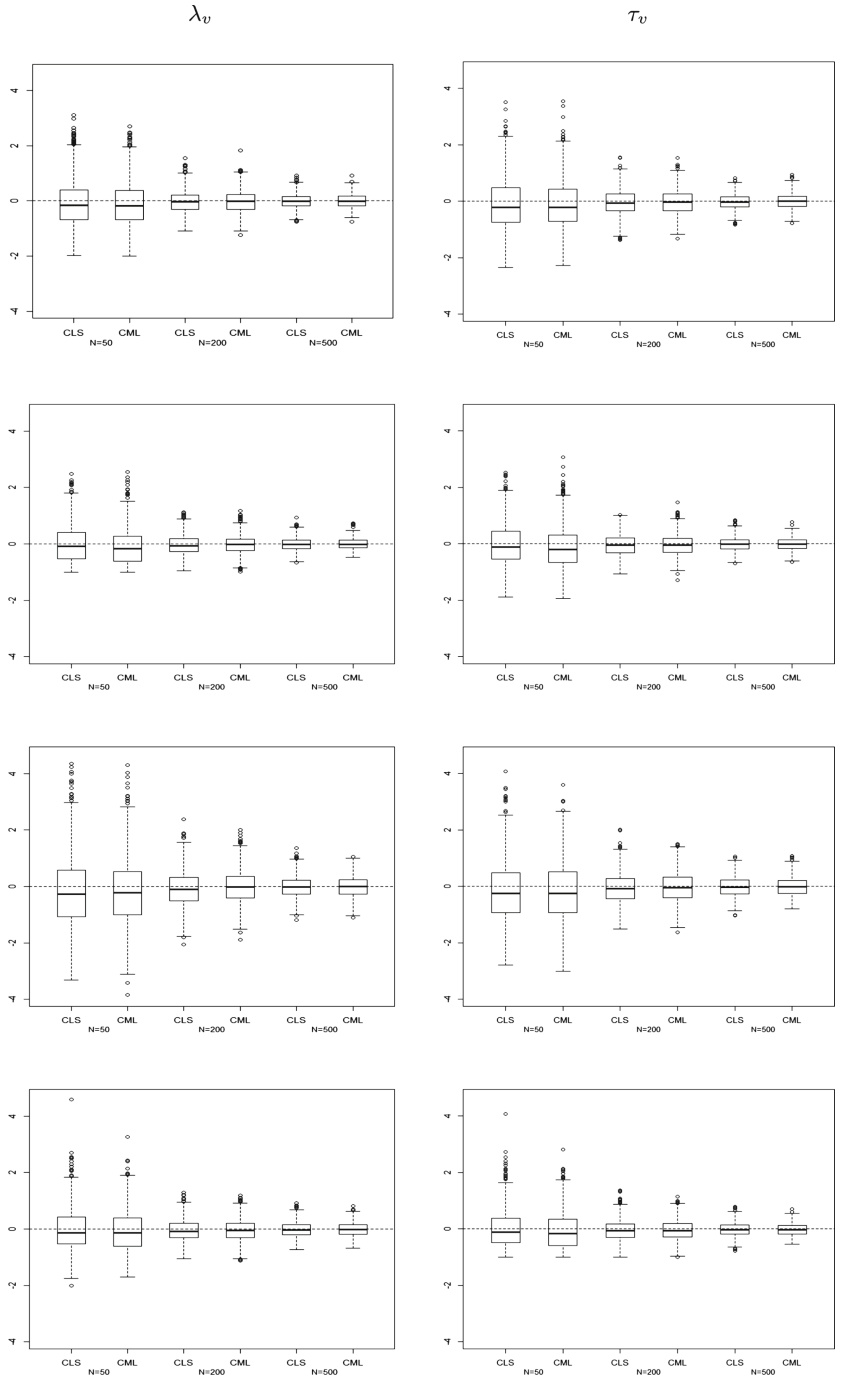


Fig. 2. Boxplots for the biases of the CLS and CML estimates of parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ in Set 1 A for $n = 4N = 200, 800, 2000$.

Table 2. CLS and CML estimates for $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\tau})$ in Set 1A. MSE in parenthesis.

	$N = 50$		$N = 200$		$N = 500$	
	CLS	CML	CLS	CML	CLS	CML
$\boldsymbol{\alpha} = (0.60, 0.40, 0.75, 0.30)$						
$\hat{\alpha}_1$	0.599 (0.0003)	0.600 (0.0006)	0.601 (0.0002)	0.600 (0.0015)	0.600 (0.0006)	0.601 (0.0004)
$\hat{\alpha}_2$	0.407 (0.0113)	0.406 (0.0001)	0.399 (0.0007)	0.402 (0.0036)	0.401 (0.0002)	0.402 (0.0001)
$\hat{\alpha}_3$	0.749 (0.0160)	0.750 (0.0116)	0.747 (0.0001)	0.751 (0.0028)	0.751 (0.0007)	0.750 (0.0001)
$\hat{\alpha}_4$	0.302 (0.0003)	0.297 (0.0095)	0.301 (0.0010)	0.300 (0.0002)	0.300 (0.0001)	0.299 (0.0002)
$\boldsymbol{\lambda} = (2, 1, 6, 5)$						
$\hat{\lambda}_1$	1.915 (0.1013)	1.880 (0.8744)	1.963 (0.0048)	1.982 (0.0774)	1.994 (0.0001)	2.000 (0.0319)
$\hat{\lambda}_2$	1.001 (0.6518)	0.895 (0.5694)	0.963 (0.0099)	0.975 (0.0357)	0.989 (0.0358)	0.998 (0.0542)
$\hat{\lambda}_3$	5.806 (0.0948)	5.810 (0.1168)	5.910 (0.1250)	5.987 (0.0944)	5.985 (0.1330)	5.999 (0.1804)
$\hat{\lambda}_4$	4.964 (0.4467)	4.936 (0.3806)	4.952 (0.0203)	4.960 (0.2504)	4.975 (0.0012)	4.977 (0.0220)
$\boldsymbol{\tau} = (4, 5, 3, 1)$						
$\hat{\tau}_1$	3.893 (1.0386)	3.880 (0.9610)	3.965 (0.0930)	3.970 (0.0106)	3.991 (0.1715)	3.998 (0.0027)
$\hat{\tau}_2$	4.977 (0.7353)	4.868 (1.2057)	4.967 (0.0203)	4.968 (0.1526)	4.987 (0.0397)	4.995 (0.0037)
$\hat{\tau}_3$	2.812 (1.0748)	2.834 (0.3680)	2.937 (0.0517)	2.970 (0.0001)	2.984 (0.0200)	2.987 (0.0295)
$\hat{\tau}_4$	1.002 (0.4951)	0.915 (0.6673)	0.959 (0.0111)	0.961 (0.2119)	0.981 (0.0225)	0.977 (0.0019)

the parameters. Nevertheless, the autoregressive parameters $\boldsymbol{\alpha}$ appear to be less biased. For smaller samples, the CLS method seems to have a better performance in estimating the parameters. Computationally, there is extra work with the CML method. The accuracy of all estimation improves as the length of the time series increases. When length increases from $N = 50$ to 200, the improvement of accuracy is more obvious than when length increases from $N = 200$ to 500.

The bias of the produced estimates were used to quantify their quality. The boxplots of the bias for CLS and CML estimates of parameters $\boldsymbol{\alpha}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ in Sets 1 A and 3 A are displayed in Figs. 1, 2, 3 and 4. These figures also show the effect of sample size on the behavior of CLS and CML estimators. No matter the

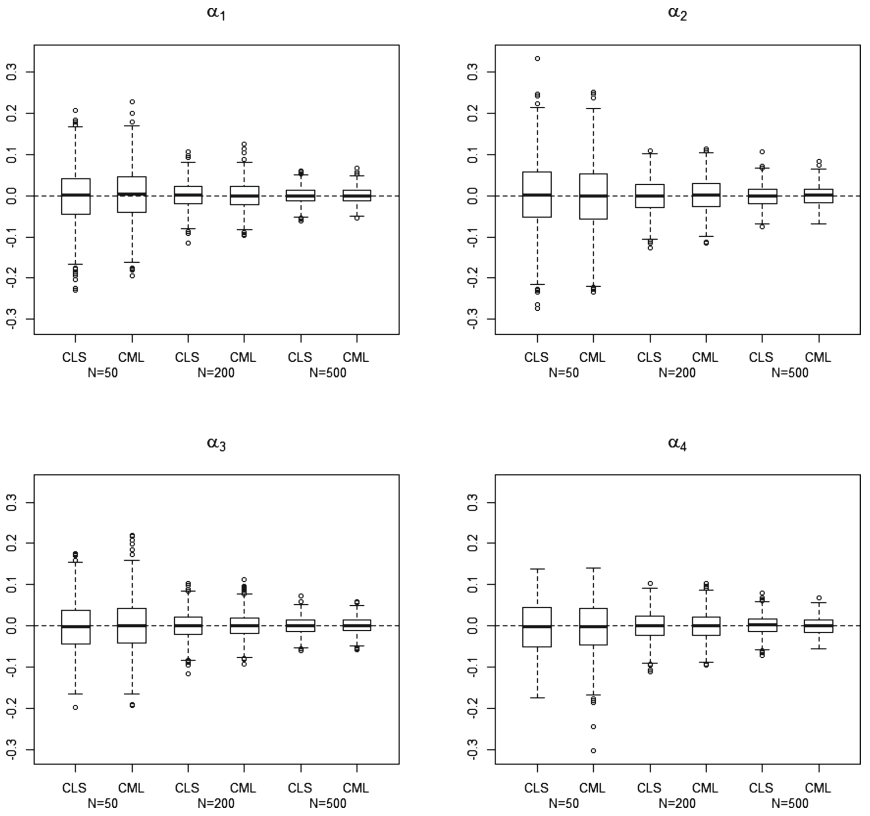


Fig. 3. Boxplots for the biases of the CLS and CML estimates of parameter $\boldsymbol{\alpha}$ in Set 3 A for $n = 4N = 200, 800, 2000$.

sample size, the difference between CLS and CML is small and becomes even smaller when the length of time series increases. The estimates for parameter $\boldsymbol{\lambda}$ seem slightly worse when parameter $\boldsymbol{\alpha}$ has all values above 0.5 (Set 3A). The results for the remaining scenarios (Sets 1B, 2A, 2B and 3B) are not presented here due to lack of space. Furthermore, Figs. 1, 2, 3 and 4 reveal that estimates of $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ componentwise tend to be biased to the left which implies that both estimation methods have a tendency to underestimate $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$, mainly in the case of small sample sizes. As expected, both bias and skewness approach zero as sample size increases. Overall, the difference between the two approaches will vanish when the length of time series increases.

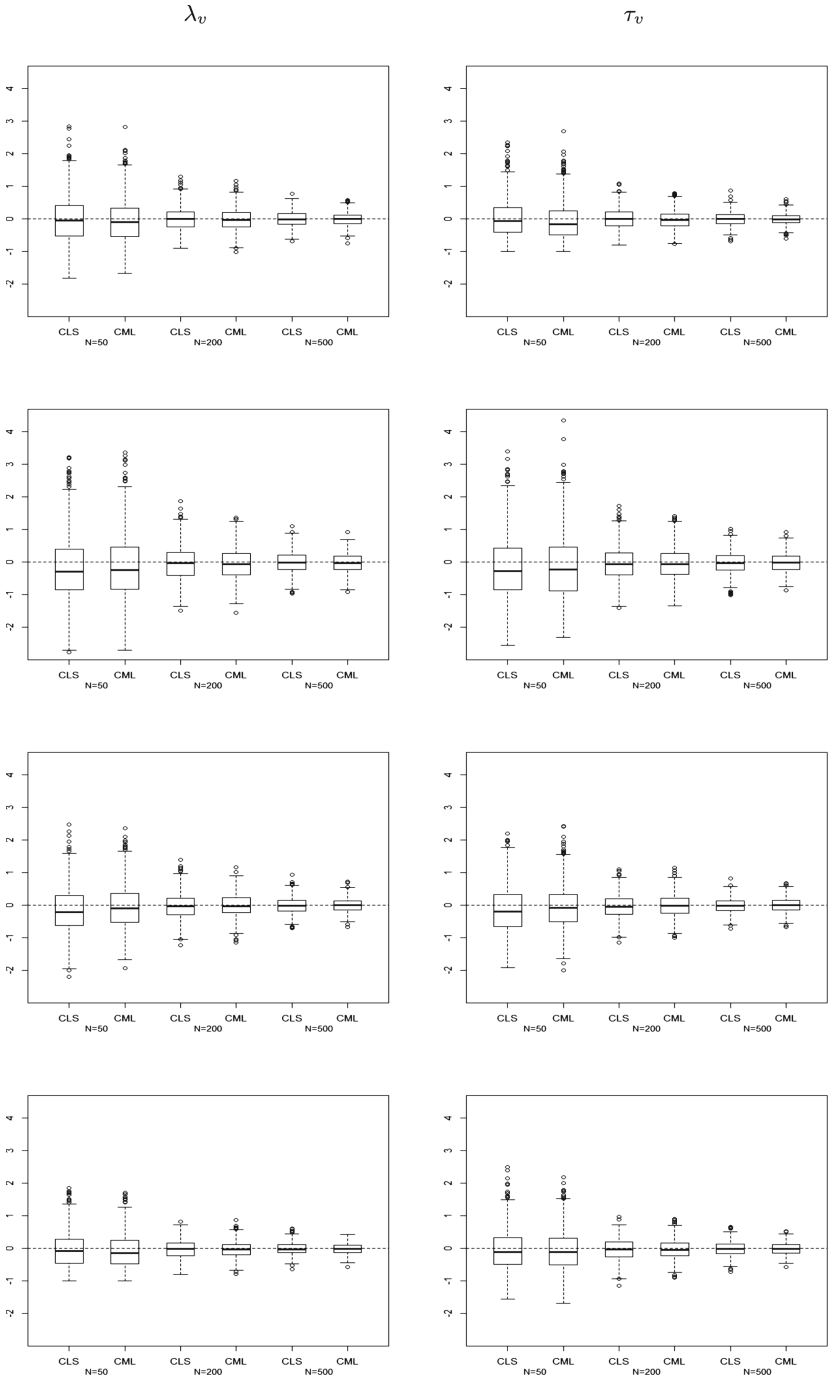


Fig. 4. Boxplots for the biases of the CLS and CML estimates of parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ in Set 3 A for $n = 4N = 200, 800, 2000$.

5 Conclusions

Our attention was focused on periodic INAR(1) models based on a different type of operator, the signed thinning operator, adapted to the periodic case. These models can handle integer-valued time series which allow for negative integer-valued and negative correlated count data unlike conventional INAR(1) models, that are only appropriate for non-negative integer-valued time series and could only deal with positive autocorrelations. Particular emphasis was given to innovations modeled by Skellam distribution defined on the set of integers. To study the performance of the conditional least squares and conditional maximum likelihood estimators, an extended simulation study was conducted for the S-PINAR(1) model with period s. Numerical results from the simulation study suggested that the proposed model is suitable for practical use. However, this is an issue we would like to explore in future work considering the application of the model to real data time series exhibiting periodic structure. Regarding periodic models based on the signed thinning operator, an important subject to investigate in further research, is the forecasting distribution of these models.

References

Alzaid, A.A., Omair, M.A.: On the poisson difference distribution inference and applications. Bull. Malays. Math. Sci. Soc. **8**(33), 17–45 (2010)

Alzaid, A.A., Omair, M.A.: An extended binomial distribution with applications. Commun. Stat.-Theory Methods **41**(19), 3511–3527 (2012)

Alzaid, A.A., Omair, M.A.: Poisson difference integer-valued autoregressive model of order one. Bull. Malays. Math. Sci. Soc. **2**(37), 465–485 (2014)

Andersson, J., Karlis, D.: A parametric time series model with covariates for integers in Z. Stat. Model. **14**(2), 135–156 (2014)

Bakouch, H.S., Kachour, M., Nadarajah, S.: An extended poisson distribution. Commun. Stat.-Theory Methods **45**(22), 6746–6764 (2016)

Chesneau, C., Kachour, M.: A parametric study for the first-order signed integer-valued autoregressive process. J. Stat. Theory Pract. **6**(4), 760–782 (2012)

Kachour, M., Truquet, L.: A p-order signed integer-valued autoregressive (SINAR(p)) model. J. Time Ser. Anal. **2**(3), 223–236 (2011)

Karlis, D., Ntzoufras, I.: Bayesian modelling of football outcomes: using the Skellam's distribution for the goal difference. J. Manag. Math. **20**(2), 133–145 (2009)

Kim, H., Park, Y.: A non-stationary integer-valued autoregressive model. Stat. Papers **49**(3), 485–502 (2008)

Klimko, L.A., Nelson, P.I.: On conditional least squares estimation for stochastic processes. Ann. Stat. **6**(3), 629–642 (1978)

Skellam, J.G.: The frequency distribution of the difference between two poisson variates belonging to different populations. J. Royal Stat. Soc. **109**(3), 296 (1946)

Steutel, F.W., van Harn, K.: Discrete analogues of self-decomposability and stability. Ann. Probab. **7**(5), 893–899 (1979)