Computational Peculiarities of the Method of Initial Functions

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Abstract. The paper investigates the computational features of the method of initial functions. Its idea is to express the components of the stress and strain state of an elastic body through initial functions defined on the initial line (a 2D problem) or surface (a 3D problem). A solution by the method of initial functions for a linear-elastic orthotropic rectangle under plane deformation is constructed. Its implementation when initial functions are represented by trigonometric functions is given. The influence of the value of a load harmonic on stable computations is studied on the example of bending of a free-supported rectangle of average thickness under the normal load specified on its upper boundary face. The causes of computational instability of the algorithm of the method of initial functions are found out. A modified algorithm is presented to increase twice the limit value of the "stable" harmonic. It is noted that calculations with a long mantissa should be cardinally performed to solve the problem of unstable computations. The results of computational experiments to determine the maximum harmonics for stable calculations of orthotropic rectangle depending on its relative thickness and mantissa length are presented. Implementation of the algorithm of the initial function method and calculations are performed using the system of analytical calculations Maple.

Keywords: Computational instability · Method of initial functions · Orthotropic solid

1 Introduction

Currently, computational methods of mechanics are the main tool for modeling the behavior of a complex mechanical system. Analytical approaches make it possible to obtain solutions as the expansion in series of functions most often for constructions of a simple configuration: a rectangle, a circle, a parallelepiped, a sphere, etc. For verification of numerical solutions to have such analytical solutions even for bodies of simple configuration, but made of materials with complex physicomechanical properties, is a good deal. Finding such solutions requires not only a high mathematical culture, but sometimes even a simple guess about the type of solution. However, for some types of configurations,

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there is a universal algorithm for constructing an analytical solution in the form of trigonometric series – a method of initial functions (MIF).

The history of this method begins with the work of Lur'e [1], in which he proposed a symbolic way of recording the solution of a system of partial differential equations in displacements of the elasticity theory. The essence of his approach was that the system of partial differential equations was considered as a system of ordinary differential equations with respect to one of the variables, and the differentiation operators with respect to other variables were simply considered as some symbolic parameters. The solution of this system was obtained as a linear combination of the unknown functions and their derivatives, defined on the initial plane $z = 0$, using singular operators in other variables. In his next works [2,3], using the solution obtained, he solves the problem of an infinite isotropic layer loaded on the two bounding surfaces $z = const$, connecting the unknown displacements and their derivatives on the initial plane with the loads specified on these surfaces.

This approach to constructing a solution was not yet the method of initial functions. The latter was born in the work of Vlasov [4], in which he, using the symbolic method of recording the solution of A.I. Lur'e, proposed a method for constructing the solution of the mixed equations of elasticity theory directly through the components of the stress-strain state defined on the initial plane: displacements u_0, v_0, w_0 and stresses σ_z^0, τ_{yz}^0, τ_{xz}^0. In the matrix-operator form, this solution can be written as:

$$\mathbf{U} = \mathbf{L}\mathbf{U}^0.$$

Here $\mathbf{U} = \{u, v, w, \sigma_z, \tau_{yz}, \tau_{xz}, \sigma_x, \sigma_y, \tau_{xy}\}$ is a vector of the stress-strain state components of the layer, $\mathbf{U}^0 = \{u_0, v_0, w_0, \sigma_z^0, \tau_{yz}^0, \tau_{xz}^0\}$ is a vector of initial functions defined on the initial plane $z = 0$ and $\mathbf{L}$ is a matrix of operators of the method of initial functions. In the literature it is called the basic relation of the method of initial functions.

After the works of Vlasov [5,6] the method of initial functions becomes very popular among a large community of scientists. The basic relations of the method of initial functions was obtained for the plane problem of elasticity theory in a rectangular orthogonal coordinate system [7–10], for the axisymmetric problem of elasticity theory in a cylindrical orthogonal coordinate system for both an isotropic continuum and an orthotropic one [11–14]. The problems of bending multilayer and layered systems with isotropic and orthotropic layers was solved [15–19]. The method of initial functions was useful in the construction of refined theories of bending of high beams and thick plates [20,21].

The method of initial functions is a universal method for constructing a solution in any canonical region of the corresponding coordinate system. The implementation of the method by trigonometric series can be carried out only in the form of a computer program. However, in this connection, it should be noted that one feature of the solutions obtained by the method of initial functions is their computational instability. To achieve satisfactory accuracy for practice, it is sometimes necessary to keep members with large harmonics in rows. But the

calculations by the initial function method at the same time can be performed with a large error. It depends on the geometric parameters of the calculated area, for example, the ratio of the sides of the rectangle or parallelepiped, and physicomechanical characteristics of the material. The natural way to eliminate the computational instability of an algorithm is to perform calculations with long mantissa. The then level of development of both computer technology and programming environments did not allow to overcome this drawback of the initial method functions, which did not allow it to establish itself widely in computational practice. But attempts to develop stable algorithms were made [22].

Modern computer technology with its high performance and new programming systems, which make it easy to go to calculations with different mantissa lengths (Maple, Python, Mathematica), makes it possible to implement computational stable algorithm of the initial function method. In this paper the results of the study of computational stability of the initial functions method based on the implementation of the algorithm of the method of initial functions in the system of analytical calculations Maple are presented. All computational experiments were performed on the example of the analysis of an orthotropic free-supported rectangle.

2 Theoretical Model

Let's consider a linearly elastic orthotropic rectangle in Cartesian orthogonal coordinate system Oxy under plane deformation (see Fig. 1). On the initial line ($x = 0$) the loads $\sigma_x^p = q\sin(py)$, $\tau_{xy}^p = 0$, $p = \dfrac{m\pi}{A}$ (m is an integer, a harmonic of the load) are specified.

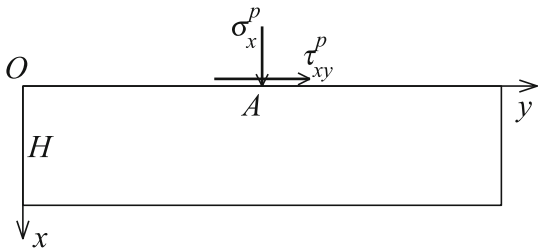

Fig. 1. A design scheme of an orthotropic rectangle.

In the case of plane deformation the equilibrium equations in displacements for an orthotropic body in a Cartesian orthogonal coordinate system Oxy in the absence of mass forces in the operator form can be written as

$$\begin{aligned}\left(A_{1,1}\partial_x^2 + A_{6,6}\partial_y^2\right) u\,(x,y) + \left(A_{1,2} + A_{6,6}\right)\partial_x\partial_y v\,(x,y) = 0,\\ \left(A_{1,2} + A_{6,6}\right)\partial_x\partial_y u\,(x,y) + \left(A_{6,6}\partial_x^2 + A_{2,2}\partial_y^2\right) v\,(x,y) = 0.\end{aligned} \tag{1}$$

Here $u(x,y), v(x,y)$ are the displacements along the axis Ox and Oy respectively, $A_{i,j}$ are elastic constants of the material and symbols ∂_x and ∂_y are differential operators with respect to the variables x and y.

According to the algorithm of the MIF the Eq. (1) are considered as ordinary differential equations with respect to the variable x while the differential operator ∂_y is assumed to be a symbolic parameter. The solution of the system of these equations is searched in the form of linear combinations of the displacements and the normal and tangential stresses defined on the initial line $x = 0$ (so called initial functions)

$$\begin{aligned} u(x,y) &= L_{1,1}(\partial_y, x)\, u^0(y) + L_{1,2}(\partial_y, x)\, v^0(y) + \\ &\quad + L_{1,3}(\partial_y, x)\, \sigma^0(y) + L_{1,4}(\partial_y, x)\, \tau^0_{xy}(y), \\ v(x,y) &= L_{2,1}(\partial_y, x)\, u^0(y) + L_{2,2}(\partial_y, x)\, v^0(y) + \\ &\quad + L_{2,3}(\partial_y, x)\, \sigma^0(y) + L_{2,4}(\partial_y, x)\, \tau^0_{xy}(y). \end{aligned} \tag{2}$$

Here $L_{ij}(\partial_y, x)$ $(i = 1,2,\ j = 1,\ldots,4)$ are unknown operator-functions (the MIF operators) and $u^0(y)$, $v^0(y)$, $\sigma^0_x(y)$, $\tau^0_{xy}(y)$ are independent and arbitrary initial functions.

Substitution (2) to (1) makes it possible to obtain the four systems of two ordinary differential equations for determining the unknown operator-functions $L_{ij}(\partial_y, x)$

$$\begin{aligned} &\left(A_{1,1}\frac{d^2}{dx^2} + A_{6,6}\partial_y^2\right) L_{1,j} + (A_{1,2} + A_{6,6})\,\partial_y \frac{d}{dx} L_{2j} = 0, \\ &(A_{1,2} + A_{6,6})\,\partial_y \frac{d}{dx} L_{1,j} + \left(A_{2,2}\partial_y^2 + A_{6,6}\frac{d^2}{dx^2}\right) L_{2,j} = 0, \quad j = 1,\ldots,4. \end{aligned} \tag{3}$$

To solve the systems (3) requires to know the initial values of the MIF operators: their values and the values of their first derivatives at the point $x = 0$. Assuming in the representation (2) $x = 0$ the values of the operators on the initial line are easily obtained

$$L_{i,j}(0) = \delta_i^j \quad (i = 1,2, j = 1,\ldots,4). \tag{4}$$

In (4) δ_i^j is the Kronecker delta.

To get the values for derivatives with respect to the variable x we use the Hooke's generalized law

$$\begin{aligned} \sigma_x &= A_{1,1}\varepsilon_x + A_{1,2}\varepsilon_y, \\ \tau_{xy} &= A_{6,6}\varepsilon_{xy}, \\ \sigma_y &= A_{1,2}\varepsilon_x + A_{2,2}\varepsilon_y \end{aligned} \tag{5}$$

and the Cauchy relations

$$\begin{aligned} \varepsilon_x &= \frac{d}{dx}u, \\ \varepsilon_x &= \partial_y v, \\ \varepsilon_{xy} &= \frac{d}{dx}v + \partial_y u. \end{aligned} \tag{6}$$

Substitution (6) into (5) gives the representation of the stress components through the displacements

$$\begin{aligned}\sigma_x &= A_{1,1}\frac{d}{dx}u + A_{1,2}\partial_y v,\\ \tau_{xy} &= A_{6,6}\frac{d}{dx}v + A_{6,6}\partial_y u,\\ \sigma_y &= A_{1,2}\frac{d}{dx}u + A_{2,2}\partial_y v.\end{aligned} \tag{7}$$

Evaluating first two equations in (7) at $x = 0$ and taking into account the representations (2) the required values of derivatives of the MIF operators are obtained as

$$\begin{aligned}&\left.\frac{dL_{1,1}}{dx}\right|_{x=0} = 0,\ \left.\frac{dL_{1,2}}{dx}\right|_{x=0} = -\frac{A_{1,2}}{A_{1,1}}\partial_y,\ \left.\frac{dL_{1,3}}{dx}\right|_{x=0} = \frac{1}{A_{1,1}},\ \left.\frac{dL_{1,4}}{dx}\right|_{x=0} = 0,\\ &\left.\frac{dL_{2,1}}{dx}\right|_{x=0} = -\partial_y,\ \left.\frac{dL_{2,2}}{dx}\right|_{x=0} = 0,\ \left.\frac{dL_{2,3}}{dx}\right|_{x=0} = 0,\ \left.\frac{dL_{2,4}}{dx}\right|_{x=0} = \frac{1}{A_{6,6}}.\end{aligned} \tag{8}$$

The solution of the systems (3) with initial values (4) and (8) gives the following values of the MIF operator-functions $L_{i,j}$ $(i = 1, 2,\ j = 1, \ldots .4)$

$$\begin{aligned}L_{1,1} &= \left[\left(-A_{1,1}A_{2,2} + A_{1,2}^2 + A_{1,2}A_{6,6} + A_{1,1}A_{6,6}\alpha_1^2\right)\cos\left(\alpha_1\partial_y x\right)\right.\\ &\quad \left. + \left(A_{1,1}A_{2,2} - A_{1,2}^2 - A_{1,2}A_{6,6} - A_{1,1}A_{6,6}\alpha_2^2\right)\cos\left(\alpha_2\partial_y x\right)\right]/d,\\ L_{1,2} &= A_{6,6}\left[\alpha_2\left(-A_{2,2} - A_{1,2}\alpha_1^2\right)\sin\left(\alpha_1\partial_y x\right)\right.\\ &\quad \left. +\alpha_1\left(A_{2,2} + A_{1,2}\alpha_2^2\right)\sin\left(\alpha_2\partial_y x\right)\right]/d,\\ L_{1,3} &= \left[\alpha_2\left(-A_{2,2} + A_{6,6}\alpha_1^2\right)\sin\left(\alpha_1\partial_y x\right)\right.\\ &\quad \left. +\alpha_1\left(A_{2,2} - A_{6,6}\alpha_2^2\right)\sin\left(\alpha_2\partial_y x\right)\right]/(\partial_y\alpha_1\alpha_2 d),\\ L_{1,4} &= \left(A_{1,2} + A_{6,6}\right)\left[\cos\left(\alpha_1\partial_y x\right) - \cos\left(\alpha_2\partial_y x\right)\right]/(\partial_y d),\\ L_{2,1} &= A_{6,6}\left[\alpha_2\left(-A_{1,2} - A_{1,1}\alpha_1^2\right)\sin\left(\alpha_1\partial_y x\right)\right.\\ &\quad \left. +\alpha_1\left(A_{1,2} + A_{1,1}\alpha_2^2\right)\sin\left(\alpha_2\partial_y x\right)\right]/(\alpha_1\alpha_2 d),\\ L_{2,2} &= A_{6,6}\left[\left(A_{1,2} + A_{1,1}\alpha_1^2\right)\cos\left(\alpha_1\partial_y x\right) + \left(-A_{1,2} - A_{1,1}\alpha_2^2\right)\cos\left(\alpha_2\partial_y x\right)\right]/d,\\ L_{2,3} &= \left(A_{1,2} + A_{6,6}\right)\left[\cos\left(\alpha_1\partial_y x\right) - \cos\left(\alpha_2\partial_y x\right)\right]/(\partial_y d),\\ L_{2,4} &= \left[\alpha_2\left(-A_{6,6} + A_{1,1}\alpha_1^2\right)\sin\left(\alpha_1\partial_y x\right)\right.\\ &\quad \left. +\alpha_1\left(A_{6,6} - A_{1,1}\alpha_2^2\right)\sin\left(\alpha_2\partial_y x\right)\right]/(\partial_y\alpha_1\alpha_2 d).\end{aligned} \tag{9}$$

Here $\alpha_1 = \sqrt{\frac{2d_4}{d_2 - d}}$, $\alpha_2 = \sqrt{\frac{2d_4}{d_2 + d}}$, $d = \sqrt{d_2^2 - 4d_0 d_4}$, $d_0 = A_{2,2}A_{6,6}$, $d_2 = A_{1,1}A_{2,2} - 2A_{1,2}A_{6,6} - {A_{1,2}}^2$, $d_4 = A_{1,1}A_{6,6}$.

Now the operator representations (2) of the displacements with the operators (9) are fully defined. To use them for analysing elastic structures the type of initial functions should be specified and then the results of the impact of the MIF operator-functions on them should be calculated.

Let the initial functions are chosen as trigonometric functions $u^0 = \overline{u}^0 \sin(py)$, $v^0 = \overline{v}^0 \cos(py)$, $\sigma_x^0 = \overline{\sigma}_x^0 \sin(py)$, $\tau_{xy}^0 = \overline{\tau}_{xy}^0 \cos(py)$, $p = m\pi/A$ and the overlined coefficients are real constants. It should define how the MIF operator-functions impact on the corresponding trigonometric functions:

$$(L_{i,1}(\partial y, x))\sin(py), (L_{i,2}(\partial y, x))\cos(py), (L_{i,3}(\partial y, x))\sin(py),$$
$$(L_{i,4}(\partial y, x))\cos(py), \quad i = 1, 2.$$

To determine the result of impacting of the MIF operator on any function this operator should be expanded in the power series. Each member of these series will contain integer power of the symbol ∂_y. Then it should perform a differentiation operation on the variable y of the function so many times what is the degree of the symbol ∂_y. Let's show how an operator $\sin(\alpha\partial_y x)$ impacts the function $\sin(py)$:

$$\begin{aligned}
(\sin(\alpha\partial_y x))\sin(py) &= \left(\sum_{i=0}^{\infty} \frac{(-1)^i(\alpha\partial_y x)^{2i+1}}{(2i+1)!}\right)\sin(py) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i(\alpha x)^{2i+1}}{(2i+1)!}\frac{d^{2i+1}\sin(py)}{dy^{2i+1}} \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i(\alpha x)^{2i+1}}{(2i+1)!}(-1)^i(p)^{2i+1}\cos(py) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^{2i}(\alpha p x)^{2i+1}}{(2i+1)!}\cos(py) = \sinh(\alpha p x)\cos(py).
\end{aligned}$$

Using this technique, the results of the impact of all transcendental operators on the trigonometric functions that compose the MIF operators are presented below

$$\sin(\alpha\partial_y x)(\sin(py)) = \sinh(\alpha p y)\cos(py),$$
$$\sin(\alpha\partial_y x)(\cos(py)) = -\sinh(\alpha p y)\sin(py),$$

$$\partial_y \sin(\alpha\partial_y x)(\sin(py)) = -p\sinh(\alpha p x)\sin(py),$$
$$\partial_y \sin(\alpha\partial_y x)(\cos(py)) = -p\sinh(\alpha p x)\cos(py),$$

$$\frac{\sin(\alpha\partial_y x)}{\partial_x}(\sin(py)) = \frac{\sinh(\alpha p x)}{p}\sin(py),$$
$$\frac{\sin(\alpha\partial_y x)}{\partial_x}(\cos(py)) = \frac{\sinh(\alpha p x)}{p}\cos(py),$$

$$\cos(\alpha\partial_y x)(\sin(py)) = \cosh(\alpha p x)\sin(py),$$
$$\cos(\alpha\partial_y x)(\cos(py)) = \cosh(\alpha p x)\cos(py),$$

$$\partial_y \cos(\alpha \partial_y x)(\sin(py)) = p\cosh(\alpha px)\cos(py),$$
$$\partial_y \cos(\alpha \partial_y x)(\cos(py)) = -p\cosh(\alpha px)\sin(py).$$

After evaluating the impact of the MIF operators on the corresponding trigonometric functions the displacements u and v will be obtained as expressions depending on hyperbolic functions. Using the Cauchy relations (6), the stress components will be also expressed through initial functions. Finally the MIF solution can be written in the matrix form as

$$\mathbf{U} = \mathbf{TL}\overline{\mathbf{U}}^0. \tag{10}$$

In (10) $\mathbf{U} = \{u, v, \sigma_x, \sigma_y, \tau_{xy}\}$ is the vector of the stress-strain state (SSS) components, $\overline{\mathbf{U}}^0 = \{\overline{u}^0, \overline{v}^0, \overline{\sigma}_x^0, \overline{\tau}_{xy}^0\}$ is the vector of the coefficients of the initial functions, $\mathbf{T} = \lceil \sin(py), \cos(py), \sin(py), \sin(py), \cos(py) \rfloor$ is a diagonal matrix of dimensions (5×5) and $\mathbf{L}$ is a matrix of dimensions (5×4) with following elements:

$$L_{1,1} = L_{3,3} = \left[\left(A_{1,1}A_{6,6}{\alpha_1}^2 - A_{1,1}A_{2,2} + {A_{1,2}}^2 + A_{1,2}A_{6,6}\right)\cosh(\alpha_1 px) + \left(-A_{1,1}A_{6,6}{\alpha_2}^2 + A_{1,1}A_{2,2} - {A_{1,2}}^2 - A_{1,2}A_{6,6}\right)\cosh(\alpha_2 px)\right]/d,$$

$$L_{1,2} = -L_{4,3} = -A_{6,6}\left[\alpha_2\left(-A_{1,2}{\alpha_1}^2 - A_{2,2}\right)\sinh(\alpha_1 px) + \alpha_1\left(A_{1,2}{\alpha_2}^2 + A_{2,2}\right)\sinh(\alpha_2 px)\right]/(\alpha_1\alpha_2 d),$$

$$L_{1,3} = \left[\alpha_2\left(A_{6,6}{\alpha_1}^2 - A_{2,2}\right)\sinh(\alpha_1 px) + \alpha_1\left(-A_{6,6}{\alpha_2}^2 + A_{2,2}\right)\sinh(\alpha_2 px)\right]/(\alpha_1\alpha_2 d),$$

$$L_{1,4} = -L_{2,3} = \left[\left(A_{1,2} + A_{6,6}\right)\left(\cosh(\alpha_1 px) - \cosh(\alpha_2 px)\right)\right]/(pd),$$

$$L_{2,1} = -L_{3,4} = A_{6,6}\left[\alpha_2\left(-A_{1,1}{\alpha_1}^2 - A_{1,2}\right)\sinh(\alpha_1 px) + \alpha_1\left(A_{1,1}{\alpha_2}^2 + A_{1,2}\right)\sinh(\alpha_2 px)\right]/(\alpha_1\alpha_2 d),$$

$$L_{2,2} = L_{4,4} = A_{6,6}\left[\left(A_{1,1}{\alpha_1}^2 + A_{1,2}\right)\cosh(\alpha_1 px) + \left(-A_{1,1}{\alpha_2}^2 - A_{1,2}\right)\cosh(\alpha_2 px)\right]/d,$$

$$L_{2,4} = \left[\alpha_2\left(A_{1,1}{\alpha_1}^2 - A_{6,6}\right)\sinh(\alpha_1 px) + \alpha_1\left(-A_{1,1}{\alpha_2}^2 + A_{6,6}\right)\sinh(\alpha_2 px)\right]/(\alpha_1\alpha_2 d),$$

$$L_{3,1} = -p\left\{\alpha_2\left[-A_{1,2}{}^2A_{6,6} + \alpha_1{}^2\left(A_{1,1}{}^2A_{2,2} - A_{1,1}A_{1,2}{}^2 - 2\,A_{1,1}A_{1,2}A_{6,6}\right) - \right.\right.$$
$$\left. + \alpha_1{}^4A_{1,1}{}^2A_{6,6}\right]\sinh(\alpha_1 px)$$
$$+\alpha_1\left[A_{1,2}{}^2A_{6,6} + \alpha_2{}^2\left(-A_{1,1}{}^2A_{2,2} + A_{1,1}A_{1,2}{}^2 + 2\,A_{1,1}A_{1,2}A_{6,6}\right)\right.$$
$$\left.\left. + \alpha_2{}^4A_{1,1}{}^2A_{6,6}\right]\sinh(\alpha_2 px)\right\}/(\alpha_1\alpha_2 d),$$

$$L_{3,2} = -L_{4,1} = -pA_{6,6}\left(-A_{1,1}A_{2,2} + A_{1,2}{}^2\right)\left(\cosh(\alpha_1 px) - \cosh(\alpha_2 px)\right)/d,$$

$$L_{4,2} = -pA_{6,6}{}^2\left[\alpha_2\left(-A_{1,1}\alpha_1{}^4 - 2\,A_{1,2}\alpha_1{}^2 - A_{2,2}\right)\sinh(\alpha_1 px)\right.$$
$$\left.+\alpha_1\left(A_{1,1}\alpha_2{}^4 + 2\,A_{1,2}\alpha_2{}^2 + A_{2,2}\right)\sinh(\alpha_2 px)\right]/(\alpha_1\alpha_2 d),$$

$$L_{5,1} = -p\left\{\alpha_2\left[-A_{1,2}A_{2,2}A_{6,6}\right.\right.$$
$$+\alpha_1{}^2\left(A_{1,1}A_{1,2}A_{2,2} - A_{1,1}A_{2,2}A_{6,6} - A_{1,2}{}^3 - A_{1,2}{}^2A_{6,6}\right)$$
$$\left.- A_{1,1}A_{1,2}A_{6,6}\alpha_1{}^4\right]\sinh(\alpha_1 px)$$
$$+\alpha_1\left[A_{1,2}A_{2,2}A_{6,6} + \alpha_2{}^2\left(-A_{1,1}A_{1,2}A_{2,2} + A_{1,1}A_{2,2}A_{6,6} + A_{1,2}{}^3 + A_{1,2}{}^2A_{6,6}\right)\right.$$
$$\left.\left.+A_{1,1}A_{1,2}A_{6,6}\alpha_2{}^4\right]\sinh(\alpha_2 px)\right\}/(\alpha_1\alpha_2 d),$$

$$L_{5,2} = -pA_{6,6}\left(A_{1,1}A_{2,2} - A_{1,2}{}^2\right)\left(\alpha_1{}^2\cosh(\alpha_1 px) - \alpha_2{}^2\cosh(\alpha_2 px)\right)/d,$$

$$L_{5,3} = A_{6,6}\left(\left(A_{1,2}\alpha_1{}^2 + A_{2,2}\right)\cosh(\alpha_1 px) + \left(-A_{1,2}\alpha_2{}^2 - A_{2,2}\right)\cosh(\alpha_2 px)\right)/d,$$

$$L_{5,4} = -\left[\alpha_2\left(-A_{2,2}A_{6,6} + \left(A_{1,1}A_{2,2} - A_{1,2}{}^2 - A_{1,2}A_{6,6}\right)\alpha_1{}^2\right)\sinh(\alpha_1 px)\right.$$
$$\left.+\ \alpha_1\left(A_{2,2}A_{6,6} + \left(-A_{1,1}A_{2,2} + A_{1,2}{}^2 + A_{1,2}A_{6,6}\right)\alpha_2{}^2\right)\sinh(\alpha_2 px)\right]/(\alpha_1\alpha_2 d).$$

How to analyse the mechanical problem with the MIF solution (10)? Usually the initial line is associated with one of the boundary line of a solid (see Fig. 1). In this case two of the initial functions become equal specified boundary conditions. Two unknown initial functions can be found by satisfying boundary conditions on the boundary line $x = H$. For this with the solution (10) the corresponding SSS components are evaluated on this boundary and equate to the boundary condition to get a system to find unknown initial functions. We will call this technique the first MIF algorithm. In our problem two initial functions σ_x^0, τ_{xy}^0 are known. Two unknown initial functions can be found from the system

$$\begin{aligned} L_{3,1}(H)\overline{u}^0 + L_{3,2}(H)\overline{v}^0 &= -\left(L_{3,3}(H)\overline{\sigma}_x^0 + L_{3,4}(H)\overline{\tau}_{xy}^0\right), \\ L_{5,1}(H)\overline{u}^0 + L_{5,2}(H)\overline{v}^0 &= -\left(L_{5,3}(H)\overline{\sigma}_x^0 + L_{5,4}(H)\overline{\tau}_{xy}^0\right). \end{aligned} \tag{11}$$

Note that on the vertical lines $y = 0, A$ of the rectangle the free support boundary conditions are satisfied: $u = 0$, $\sigma_y = 0$.

3 Investigation of the Solution and Discussion

Consider the behavior of the MIF solution (10) with increasing harmonic m of normal load $\overline{\sigma}_x^0 = q \sin(m\pi y/A)$ on the upper boundary of the rectangle with no load on the bottom boundary. Let the technical constants of the orthotropic material are $E_x = 10^6$, $E_y = 25\,10^6$, $\nu_{yx} = 0.25$, $\nu_{xy} = (E_x/E_y)\ \nu_{yx}$, $G_{xy} = 5\,10^5$ and the ratio of the width of the rectangle to its height is equal $A/H = 4$. The elastic constants $A_{i,j}$ are expressed through the technical modules of elasticity as $A_{1,1} = \dfrac{E_x}{1-\nu_{xy}\nu_{yx}}$, $A_{2,2} = \dfrac{E_y}{1-\nu_{xy}\nu_{yx}}$, $A_{1,2} = \dfrac{E_x\nu_{yx}}{1-\nu_{xy}\nu_{yx}} = \dfrac{E_y\nu_{xy}}{1-\nu_{xy}\nu_{yx}}$, $A_{6,6} = G_{xy}$.

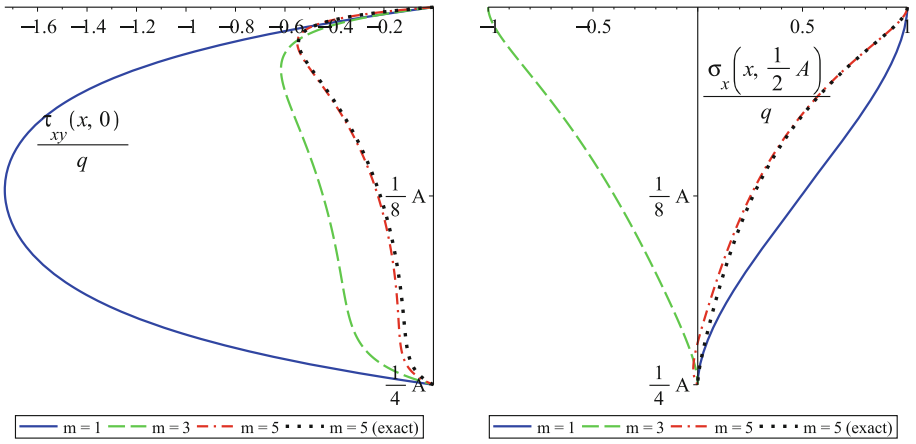

Fig. 2. Dimensionless shear stresses τ_{xy}/q (a) in the cross section $y = 0$ and normal stresses σ_x/q (b) in the cross section $y = A/2$ across the plate thickness when calculating by the first MIF algorithm. (Color figure online)

The dimensionless shear stresses τ_{xy}/q (the cross section $y = 0$) and normal stresses σ_x/q (b) (the cross section $y = A/2$) under three loads with harmonics $m = 1, 3, 5$ are presented on the Fig. 2. Calculations have been performed with double precision (16 decimal digits in the mantissa). The stresses under load with a harmonic of $m = 5$ in the neighbourhood of the bottom boundary are calculated with an error. This is evident for normal stress σ_x (the red dashdotted line in the Fig. 2b) which becomes negative in the lower part of the plate. For shear stresses, this fact is not so obvious (see the red dashdotted line in the Fig. 2a). To verify this, the Fig. 2 also presents a graph of the exact calculation of the shear stress (see the black dotted line). Minor differences start at the middle of the plate and reach the highest values at the lower boundary. With a further increase in harmonics, the region of unstable calculation expands, starting ever

closer to the upper boundary of the plate. In this case, the errors can reach enormous values, reaching up to orders of tens.

What is the reason for such a behavior of our analytical solution (10)? The answer lies in the structure of the solution. It is necessary to do the calculation of the difference between two large quantities. The normal stress $\sigma_x(x,y)$ is calculated according to the MIF solution by the formula

$$\sigma_x(x,y) = \left[L_{3,1}(x)\,\overline{u}^0 + L_{3,2}(x)\,\overline{v}^0 + L_{3,3}(x)\,\overline{\sigma}_x^0 + L_{4,4}(x)\,\overline{\tau}_{xy}^0\right]\sin\left(\frac{m\pi y}{A}\right).$$

From the boundary conditions on the upper face of the rectangle, it is determined that $\overline{\sigma}_x^0 = q$ and $\overline{\tau}_{xy}^0 = 0$ and the values of the initial displacements $\overline{u}^0$ and $\overline{v}^0$ can be found when solving the linear algebraic system (11). Thus, the formula for calculating the normal stress $\sigma_x(x,y)$ is transformed as

$$\sigma_x(x,y) = \left[L_{3,1}(x)\,\overline{u}^0 + L_{3,2}(x)\,\overline{v}^0 + L_{3,3}(x)\,q\right]\sin\left(\frac{m\pi y}{A}\right). \tag{12}$$

Consider computing with a mantissa length equal to 16 (double precision). When $m = 1$ the displacements $\overline{u}^0 = -1.33382195199\overline{7764}\,10^{-6}q$ and $\overline{v}^0 = -1.7902511821\overline{84309}\,10^{-6}q$, thus the formula (12) for evaluating the value of the stress in the point $(49H/50, A/2)$ gives following result

$$\sigma_x(49H/50, A/2) = -1.514358973839\overline{9687}\,10^7\,\overline{u}^0 + 1.125694145232\overline{843}\,10^8\,\overline{v}^0 - 0.1103210932\overline{2631}\,q.$$

Overlined digits in decimal notation represent incorrect digits that were determined by comparison with the exact calculations made with the mantissa length equal to 20. Taking into account the errors of the coefficients and the values of the initial functions in the given formula, the theoretical absolute error of the normal stress is expressed as

$$\Delta = \left[10^{-5}10^{-6} + 10^{7}10^{-16}\right] + \left[10^{-5}10^{-6} + 10^{8}10^{-16}\right] + \left[10^{-10}\right] = 10^{-9}.$$

Here in square brackets are the absolute errors of three members of the sum in (12). Substituting numerical values for $\overline{u}^0$ and $\overline{v}^0 q$ into the formula for σ_x, we get that the value of the normal stress is calculated as the difference

$$\begin{aligned}\sigma_x(49H/50, A/2) &= \left(32.0861851\overline{8321481} - 32.0845713\overline{9048015}\right)q \\ &= 0.0016137927\overline{34682}q.\end{aligned} \tag{13}$$

It should be seen that the absolute error is two orders of magnitude less than theoretical one. This is a great result. But with increasing harmonics, the coefficients in the formula and the initial functions themselves are calculated with a large error, which leads to a catastrophic loss of accuracy. For $m = 5$ we have the following results:

Table 1. Limit values of harmonics for stable calculations depending on the geometric dimensions of the orthotropic rectangle and the mantissa length for the first MIF algorithm.

A/H	Mantissa length				
	16	**24**	**32**	**40**	**48**
10	14	22	30	39	47
6	8	13	18	23	28
4	5	9	12	15	18
2	2	4	6	7	9

$$\sigma_x\,(49H/50, A/2) = -2.915263974948\overline{411}\,10^{17}\,\overline{u}^0 + 2.041089167896\overline{948}\,10^{18}\,\overline{v}^0 - 4.109354721633\overline{858}\,10^9\,q,$$

$$\overline{u}^0 = -9.\overline{648117714642594}\,10^{-8}q, \overline{v}^0 = -1.1\overline{76697993193913}\,10^{-8}q,$$

$$\Delta = \left[10^6 10^{-8} + 10^{17}10^{-10}\right] + \left[10^7 10^{-8} + 10^{18}10^{-11}\right] + \left[10^{-3}\right] = 10^7,$$

$$\sigma_x\,(49H/50, A/2) = \left(2.8\overline{12680999955915}\,10^{10} - 2.8\overline{12680999957558}\,10^{10}\right) q = -0.\overline{16440418280285}\,10^{-1}q.$$

We see that there are no true digits in the obtained numerical value. Moreover, even the sign is negative, which can not be from a physical point of view. Such a behavior of the constructed solution is due to the fact that as the harmonic increases, the values of the matrix components and the vector of the right-hand side of the ill-conditioned linear system (11) begin to grow. The length of the mantissa is not enough to accurately represent the integer part of a number of large order. Increasing the length of the mantissa, the MIF solution allows to obtain exact numerical values. The value of the "stable" mantissa depends on material characteristics and the ratio of geometrical dimensions of the rectangle. Table 1 shows the "stable" mantissa values for the considered orthotropic material.

A way to increase the harmonic of stable calculations with a given mantissa can be offered. It is necessary to change the coordinate system by shifting the axis Oy to a point located in the middle of the height of the rectangle. In this case, the initial functions will no longer be equal to the given boundary conditions. Four equations will be required to find all four unknown initial functions

$$L_{3,1}(-H/2)\overline{u}^0 + L_{3,2}(-H/2)\overline{v}^0 + L_{3,3}(-H/2)\overline{\sigma}_x^0 + L_{3,4}(-H/2)\overline{\tau}_{xy}^0 = \sigma_x^-,$$
$$L_{4,1}(-H/2)\overline{u}^0 + L_{4,2}(-H/2)\overline{v}^0 + L_{4,3}(-H/2)\overline{\sigma}_x^0 + L_{4,4}(-H/2)\overline{\tau}_{xy}^0 = \tau_{xy}^-,$$
$$L_{3,1}(H/2)\overline{u}^0 + L_{3,2}(H/2)\overline{v}^0 + L_{3,3}(H/2)\overline{\sigma}_x^0 + L_{3,4}(H/2)\overline{\tau}_{xy}^0 = \sigma_x^+,$$
$$L_{4,1}(H/2)\overline{u}^0 + L_{4,2}(H/2)\overline{v}^0 + L_{4,3}(H/2)\overline{\sigma}_x^0 + L_{4,4}(H/2)\overline{\tau}_{xy}^0 = \tau_{xy}^+.$$

Table 2. Limit values of harmonics for stable calculations depending on the geometric dimensions of the orthotropic rectangle and the mantissa length for the second MIF algorithm.

A/H	Mantissa length				
	16	**24**	**32**	**40**	**48**
10	25	45	61	81	91
6	15	27	37	49	59
4	9	17	23	31	39
2	5	9	12	16	19

Here σ_x^+, τ_x^+ are boundary conditions on the upper rectangle's face and σ_x^-, τ_x^- are ones on the lower rectangle's face. We will call this algorithm the second MIF algorithm. Figure 2 shows stable calculations up to the eleventh harmonic (exclusively). We can also observe the difference in the calculated stresses ($m = 11$) with double precision (red dashdotted graphs) and exact results (black dotted graphs).

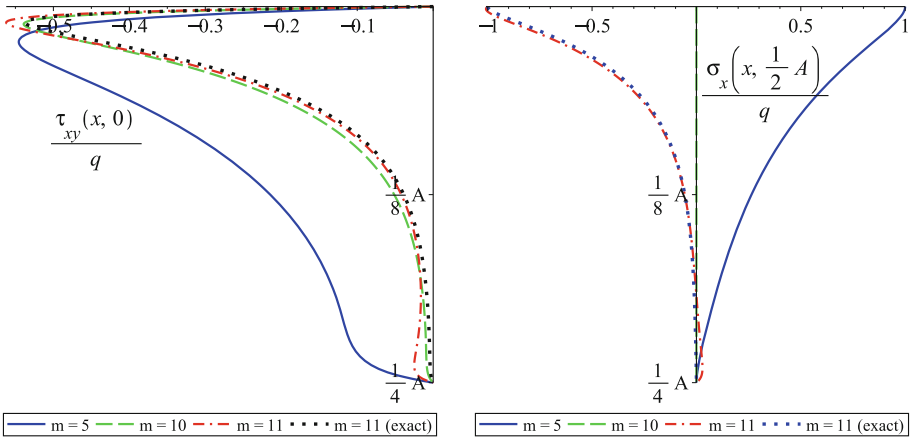

Fig. 3. Dimensionless shear stresses τ_{xy}/q (a) in the cross section $y = 0$ and normal stresses σ_x/q (b) in the cross section $y = A/2$ across the plate thickness when calculating by the second MIF algorithm.

This technique increases the limit value of the harmonic for the mantissa length because the maximum value of the coordinate x is equal $\pm H/2$ (see Table 2).

4 Conclusion

We have seen that MIF solutions of bending problems of linear-elastic orthotropic rectangular plates under boundary conditions in the form of trigonometric functions have computational instability that does not allow them to be used directly for loads represented by trigonometric series.

Computational experiments have shown that with the increase of the load harmonic the computational instability of the algorithm of the method of initial function begins to manifest itself. Starting with a certain harmonic number, calculations of the stress-strain state components in the neighbourhood of the lower face of the plate begin to be performed with a large error. This error forms from errors in the determination of unknown initial functions solving an ill-conditioned system of linear equations and errors in the representation of the coefficients of this system and its right parts having an order greater than the number of significant digits in the mantissa of the representation of real numbers.

This disadvantage of the method can be partially overcome by combining the initial line with the median line of the rectangle. The limiting values of "stable" harmonics are doubled, which is associated with a decrease in two times the coordinate values of the boundary lines of the rectangle.

But this approach does not definitively solve the problem of instability of computations. Radical results can be achieved by increasing the length of the mantissa in the representation of real numbers. Of course, this is due both to time costs and to the increase in the amount of RAM used. However, with the presence of high-speed modern computers, as well as analytical computing systems (for example, Maple) and programming languages (for example, Python) with the possibility of a simple transition to calculations with a mantissa of arbitrary length, these shortcomings do not seem to be so significant.

In conclusion, we note that the presented investigation of the stability of the method of initial functions were used in computational algorithms based on a common solution for an isotropic/orthotropic rectangle. This solution was obtained by the superposition method as the sum of two (for the plane problem) and three (for the spatial problem) solutions obtained by the method of initial functions in each of the independent coordinates. With its help, the problems of bending a grillage [23] and an isotropic/orthotropic rectangle [24] with arbitrary boundary conditions on all their faces were investigated. For the plane problem of the theory of elasticity, a method of analytical decomposition has been developed on the basis of the general solution for analysing complex inhomogeneous elastic systems [25].

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