

Minimizing the Energy of a Quad Rotor in Free Final Time Using Bocop Software

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Abstract. An unmanned aerial vehicle (UAV) become increasingly present in military and commercial applications, the flight path efficiency and integration with current manned aircraft become important research topics to address in the coming years. In this paper, the optimal control problem for the quad-rotor to minimize energy is considered. For this, formulate this problem to an optimal control problem which minimizes energy of quad-rotor in free final time. The Solution for this, used a discretization method, and simulation results are given by Bocop software.

Keywords: Final time · Bocop · Optimal control

1 Introduction

An Unmanned Aerial Vehicle is an aircraft with no pilot on board [1–3]. UAVs can be remote controlled aircraft (e.g. flown by a pilot at a ground control station) or can fly autonomously based on pre-programmed flight plans or more complex dynamic automation systems. UAVs are currently used for a number of missions, including reconnaissance and attack roles. For the purposes of this article, and to distinguish UAVs from missiles, a UAV is defined as being capable of controlled, sustained level flight and powered by a jet or reciprocating engine. In addition, a cruise missile can be considered to be a UAV, but is treated separately on the basis that the vehicle is the weapon. The acronym UAV has been expanded in some cases to UAVS (Unmanned Aircraft Vehicle System). The FAA has adopted the acronym UAS (Unmanned Aircraft System) to reflect the fact that these complex systems include ground stations and other elements besides the actual air vehicles [6].

Some early UAVs are called drones because they are no more sophisticated than a simple radio controlled aircraft being controlled by a human pilot (sometimes called the operator) at all times. More sophisticated versions may have

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built-in control and/or guidance systems to perform low level human pilot duties such as speed and flight path stabilization, and simple prescripted navigation functions such as waypoint following.

From this perspective, most early UAVs are not autonomous at all. In fact, the field of air vehicle autonomy is a recently emerging field, whose economics is largely driven by the military to develop battle ready technology for the warfighter. Compared to the manufacturing of UAV flight hardware, the market for autonomy technology is fairly immature and undeveloped. Because of this, autonomy has been and may continue to be the bottleneck for future UAV developments, and the overall value and rate of expansion of the future UAV market could be largely driven by advances to be made in the field of autonomy.

Autonomy is commonly defined as the ability to make decisions without human intervention. To that end, the goal of autonomy is to teach machines to be "smart" and act more like humans. The keen observer may associate this with the development in the field of artificial intelligence made popular in the 1980s and 1990s such as expert systems, neural networks, machine learning, natural language processing, and vision. However, the mode of technological development in the field of autonomy has mostly followed a bottom-up approach, and recent advances have been largely driven by the practitioners in the field of control science, not computer science. Similarly, autonomy has been and probably will continue to be considered an extension of the controls field. In the foreseeable future, however, the two fields will merge to a much greater degree, and practitioners and researchers from both disciplines will work together to spawn rapid technological development in the area [4,5].

To some extent, the ultimate goal in the development of autonomy technology is to replace the human pilot. It remains to be seen whether future developments of autonomy technology, the perception of the technology, and most importantly, the political climate surrounding the use of such technology, will limit the development and utility of autonomy for UAV applications [26,29].

In our case, We choose quad-rotors drone which is handy, allow vertical takeoff and landing, as well as flying in hard to - Reach areas. The disadvantages are its mass and the consumption of energy caused by motors. The drone can perform three flight modes; hover, vertical flight and translation flight. In our work, we are interested in a translation flight that corresponds to the navigation on a horizontal plane, it is ensured by basing itself on pitch and roll tilting movements [25].

In the present study, our aim is to minimize energy for quadrotor in free final time. To solve this problem, let's build an optimal control problem in free final time. Using Bocop Software [28], we solve this problem with discretization method as Midpoint method in small step to insure convergence [24,27]. The work presented in this paper is organized as follows: In Sect. 1, a dynamic model of the quad-rotor is considered. In Sect. 2, discussion, simulation results given by Bocop software and conclusion are provided in Sect. 3.

2 Quad-Rotor Mathematical Model

A quad-rotor is an aerial vehicle which rotates by producing differentials in thrust between it's for motors. The following section presents the mathematical model of the quad-rotor.

The motion of the quad-rotor can be divided into two subsystems, rotational subsystem (roll, pitch and yaw) and translation subsystem (altitude and x and y position).

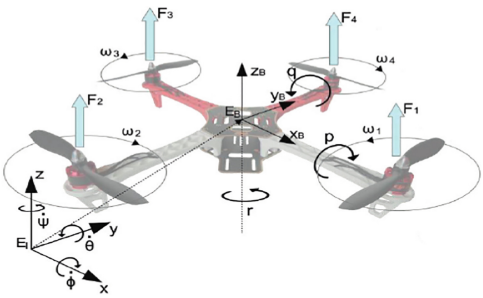

Fig. 1. Quad-rotor

2.1 Rotational Equations of Motion

The rotational equations of motion are derived in the body frame using the Newton-Euler method with the following general formalism,

$$J\dot{w} + w \times Jw \times M_G = M_B \tag{1}$$

where

- J: Quad-rotor's diagonal inertia Matrix.
- w: Angular body rates.
- M_B: Gyroscopic moments due to rotor's inertia.
- M_G: Moments acting on the quad-rotor in the body frame.
- $J\dot{w}$ and $w \times Jw$ represent the rate of change of angular momentum in the body frame.
- M_G: represent the gyroscopic moments due to the rotor's inertia J_r. The Gyroscopic moments are defined to be $w \times \begin{bmatrix} 0 & 0 & J_r w_r \end{bmatrix}^T$, thus the rotational equation of the quad-rotor's motion can be written as

$$J\dot{w} + w \times Jw + w \times \begin{bmatrix} 0 & 0 & J_r w_r \end{bmatrix}^T = M_B \tag{2}$$

where

- J_r: rotor's inertia.
- Ω : rotor's relative speed $\Omega_r = -w_1 + w_2 - w_3 + w_4$.

The inertia matrix for the quad-rotor is a diagonal matrix, the off-diagonal elements, which are the product of inertia, are zero due to the symmetry of the quad-rotor:

$$J = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{3}$$

where I_x, I_y and I_z are the area moments of inertia about the principle axes in the body frame.

Equations (4), (5) show the aerodynamique force F_i and moment M_i produced by the ith rotor $i = \overline{1,4}$

$$\begin{aligned} F_i &= K_f w_i^2 \\ M_i &= K_m w_i^2 \end{aligned} \tag{4}$$

where K_f and K_M are the aerodynamic force and moment constants respectively and w_i is the angular velocity of the rotor i.

Thus, the total moment about the $x-$ axis can be expressed as

$$\begin{aligned} M_x &= -F_2 l + F_4 l \\ &= -(K_f w_2^2) l + (K_f w_4^2) l \\ &= l K_f (-w_2^2 + w_4^2) \end{aligned} \tag{5}$$

the moment about $y-$ axis

$$\begin{aligned} M_y &= -F_1 l - F_3 l \\ &= (K_f w_1^2) l - (K_f w_3^2) l \\ &= l K_f (-w_1^2 - w_3^2) \end{aligned} \tag{6}$$

the moment about $z-$axis

$$\begin{aligned} M_z &= M_1 - M_2 + M_3 - M_4 \\ &= (K_m w_1^2) - (K_m w_2^2) + (K_m w_3^2) - (K_m w_4^2) \\ &= K_m (-w_1^2 - w_2^2 + w_3^2 - w_4^2) \end{aligned} \tag{7}$$

Combining Eqs. (5), (6) and (7) in vector form, we get,

$$M_B = \begin{bmatrix} l K_f (-w_2^2 + w_4^4) \\ l K_f (w_1^2 - w_4^4) \\ K_m (w_1^2 - w_2^2 + w_3^2 - w_4^2) \end{bmatrix} \tag{8}$$

where l is the moment arm, which is the distance between them axis of rotation each rotor to the engine for the body reference frame which should coincide with the center of the quad-rotor.

3 Translation Equations of Motion

The translation equations of the equations of the quad-rotor are based on Newton's second law and they are derived in the earth inertial frame

$$m\ddot{r} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + RF_B \tag{9}$$

- $r = [x, y, z]^T$: Quad-rotor's distance from the inertial frame.
- m: Quad-rotor's mass.
- g: gravitationnal acceleration $g = 9,81m/s^2$.
- F_B: non gravitational forces acting on the quad-rotor in the body frame

$$F_B = \begin{bmatrix} 0 \\ 0 \\ K_f(w_1^2 + w_2^2 + w_3^2 + w_4^2) \end{bmatrix} \tag{10}$$

The first two rows of the force vector are zeros as there is no forces in the X and Y directions, the last row is the thrust forces produced by the propellers.

The drag forces F_a can be given as:

$$F_a = K_t\dot{r} \tag{11}$$

where K_t is a matrix called the aerodynamic translation coefficient matrix and $\dot{r}$ is the time derivative of the position vector r.

$$m\ddot{r} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + RF_B - F_a \tag{12}$$

And Drag moment is defined as follows:

$$M_a = K_r\dot{\eta} \tag{13}$$

where K_r is a constant matrix called the aerodynamic rotation coefficient matrix and $\dot{\eta}$ is the Euler rates.

$$J\dot{w} + w \times Jw + w \times \begin{bmatrix} 0 & 0 & J_r w_r \end{bmatrix}^T = M_B - M_a \tag{14}$$

Let be defined the state vector of the quad-rotor as

$$X = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}]^T \tag{15}$$

which is mapped to the degrees of freedom of the quad-rotor in the following manner.

$$X = [\phi, \dot{\phi}, \theta; \dot{\theta}, \psi, \dot{\psi}, x, \dot{x}, y, \dot{y}, z, \dot{z}]^T \tag{16}$$

The state vector defines the position of the quad-rotor in space and its angular and linear velocities.

3.1 Control Input Vector u

A control input vector u, considering of four inputs, is defined as:

$$u = \begin{bmatrix} u_1 \; u_2 \; u_3 \; u_4 \end{bmatrix} \tag{17}$$

$$u_1 = K_f(w_1^2 + w_2^2 + w_3^2 + w_4^2) \tag{18}$$

$$u_2 = K_f(w_1^2 + w_4^2) \tag{19}$$

$$u_3 = K_f(w_1^2 - w_3^2) \tag{20}$$

$$u_4 = K_m(w_1^2 - w_2^2 + w_3^2 - w_4^2) \tag{21}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} K_f & K_f & K_f & K_f \\ 0 & -K_f & 0 & K_f \\ K_f & 0 & -K_f & 0 \\ K_m & -K_m & K_m & -K_m \end{bmatrix} \begin{bmatrix} w_1^2 \\ w_2^2 \\ w_3^2 \\ w_4^2 \end{bmatrix} \tag{22}$$

where:

- u_1: lift force.
- u_2, u_3, u_4: aerodynamical moments developed by the system.

Substituting (16)–(19) in Eq. (8), the equation of the total moments acting on the quad-rotor becomes:

$$M_B = \begin{bmatrix} lu_2 \\ lu3 \\ u_4 \end{bmatrix} \tag{23}$$

Substituting (21) into the rotational equation of motion (4), can be derived,

$$\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$
$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ J_r w_r \end{bmatrix} = \begin{bmatrix} lu_2 \\ lu_3 \\ u_4 \end{bmatrix} \tag{24}$$

Expanding that, leads to:

$$\begin{bmatrix} I_x\ddot{\varphi} \\ I_y\ddot{\theta} \\ I_y\ddot{\psi} \end{bmatrix} \begin{bmatrix} \dot{\theta}I_z\dot{\psi} - \dot{\psi}I_y\dot{\theta} \\ \dot{\psi}I_x\dot{\varphi} - \dot{\varphi}I_z\dot{\psi} \\ \dot{\theta}I_y\dot{\theta} - \dot{\theta}I_x\dot{\varphi} \end{bmatrix} + \begin{bmatrix} \dot{\theta}J_r w_r \\ -\dot{\varphi}J_r w_r \\ 0 \end{bmatrix} = \begin{bmatrix} lu_2 \\ lu_3 \\ u_4 \end{bmatrix} \tag{25}$$

Rewriting the last equation to have the angular acceleration in terms of the other variables:

$$\ddot{\varphi} = \frac{l}{I_x}u_2 - \frac{J_r}{I_x}\dot{\theta}w_r + \frac{I_y}{I_z}\dot{\psi}\dot{\theta} - \frac{I_z}{I_x}\dot{\theta}\dot{\psi} \tag{26}$$

$$\ddot{\theta} = \frac{l}{I_y}u_3 - \frac{J_r}{I_y}\dot{\varphi}w_r + \frac{I_z}{I_y}\dot{\varphi}\dot{\psi} - \frac{I_x}{I_y}\dot{\psi}\dot{\varphi} \tag{27}$$

$$\ddot{\psi} = \frac{1}{I_z}u_4 + \frac{I_x}{I_z}\dot{\theta}\dot{\psi} - \frac{I_y}{I_z}\dot{\varphi}\dot{\theta} \tag{28}$$

to simplify, define

$$a_1 = \frac{I_y - I_z}{I_x}, a_2 = \frac{J_r}{I_x}, a_3 = \frac{I_z - I_x}{I_y}, a_4 = \frac{J_r}{I_y},$$
$$a_5 = \frac{I_x - I_y}{I_z}, b_1 = \frac{l}{I_x}, b_2 = \frac{l}{I_y}, b_3 = \frac{1}{I_x}$$

Using the above notation of $a_1 \rightarrow a_5$ and $b_1 \rightarrow b_3$, Eqs. (24)–(26) can then be rewritten in a simpler form in terms of the system states

$$\ddot{\varphi} = b_1 u_2 - a_2 x_4 w_r + a_1 x_4 x_6 \tag{29}$$

$$\ddot{\theta} = b_2 u_3 + a_4 x_2 w_r + a_3 x_2 x_6 \tag{30}$$

$$\ddot{\psi} = b_3 u_4 + a_5 x_2 x_4 \tag{31}$$

Substituting equation (16)–(19) in Eq. 10, the equation of the total moments acting on the quad-rotor becomes:

$$F_B = \begin{bmatrix} 0 \\ 0 \\ -u_1 \end{bmatrix} \tag{32}$$

Embedding that into the translational equation of motion (9) and expanding the term, we get

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + \begin{bmatrix} c\psi c\theta & c\psi s\varphi s\theta & s\theta s\psi + c\varphi c\psi s\theta \\ c\theta s\psi & c\theta c\psi + s\varphi s\psi s\theta & c\varphi s\psi s\theta - c\psi s\theta \\ -s\theta & c\theta s\varphi & c\varphi c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -u_1 \end{bmatrix} \tag{33}$$

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + \begin{bmatrix} (c\varphi s\psi + c\varphi c\psi s\theta)(-u_1) & (c\theta s\psi s\theta - c\psi s\varphi)(-u_1) \\ -(c\varphi c\theta)(-u_1) \end{bmatrix} \tag{34}$$

Rewriting Eq. (32) to have acceleration in term of the other variables, we get

$$\ddot{x} = \frac{-u_1}{m}(sin\varphi sin\psi + cos\varphi cos\psi sin\theta) \tag{35}$$

$$\ddot{y} = \frac{-u_1}{m}(cos\varphi sin\psi sin\theta - cos\psi sin\varphi) \tag{36}$$

$$\ddot{z} = g - \frac{-u_1}{m}(cos\varphi cos\theta) \tag{37}$$

Rewriting in terms of the state variable X

$$\ddot{x} = \frac{-u_1}{m}(sinx_1 sinx_5 + cosx_1 cosx_5 sinx_3) \tag{38}$$

$$\ddot{y} = \frac{-u_1}{m}(cosx_1 sinx_5 sinx_3 - cosx_5 sinx_1) \tag{39}$$

$$\ddot{z} = g - \frac{-u_1}{m}(cosx_1 cosx_3) \tag{40}$$

Using the equation of the rational angular acceleration. Equations (27)–(29), and those of translation, Eqs. (35)–(37), the mathematical model of the quad-rotor can be written in a state space representation as follows

$$\begin{cases} \dot{x}_1 = \dot{\varphi} = x_2, \\ \dot{x}_2 = \ddot{\varphi} = a_1 x_4 x_6 + a_2 x_4 \Omega + b_1 u_1, \\ \dot{x}_3 = \dot{\theta} = x_4, \\ \dot{x}_4 = \ddot{\theta} = a_3 x_2 x_6 + a_4 x_2 \Omega + b_2 u_2, \\ \dot{x}_5 = \dot{\psi} = x_6, \\ \dot{x}_6 = \ddot{\psi} = a_5 x_2 x_4 + b_3 u_3, \\ \dot{x}_7 = \dot{z} = x_8, \\ \dot{x}_8 = \ddot{z} = \frac{u_4}{m}(cosx_1 sinx_3 cosx_5 + sinx_1 sinx_5), \\ \dot{x}_9 = \dot{x} = x_{10}, \\ \dot{x}_{10} = \ddot{x} = \frac{u_4}{m}(cosx_1 sinx_3 sinx_5 - sinx_1 cosx_5), \\ \dot{x}_{11} = \dot{y} = x_{12}, \\ \dot{x}_{12} = \ddot{y} = \frac{cosx_1 cosx_3}{m} u_4 - g, \\ -\pi/2 \leq x_1 \leq \pi/2, -\pi/2 \leq x_3 \leq \pi/2, \\ -\pi \leq x_5 \leq \pi, x_i(0) = 0, i = \overline{1,12}, \\ -20 \leq u_j \leq 20,\ j = \overline{1,3},\ 0 \leq u_4 \leq 20. \end{cases} \tag{41}$$

where $g(m/s^2)$: gravity acceleration; $I_x, I_y, I_z(kg/m^2)$:roll, pitch and yaw inertia moments respectively, $J_r(kg/m^2)$: the rotor inertia; $m(kg)$: mass; $x, y, z(m)$: longitudinal, lateral and vertical motions respectively; $\phi, \theta, \psi(rad)$: roll, pitch and yaw angles, respectively; $w_k(rad/s)$: rotor angular velocity, where, k equal to 1, 2, 3 and 4; $d(m)$: the distance between the quad-rotor center of mass and the propeller rotation axis; $u_1, u_2, u_3(N.m)$: aerodynamical roll, pitch and yaw moments respectively; $u_4(N)$: lift force.

With $\Omega = w_1 - w_2 + w_3 - w_4$, $a_1 = \frac{I_y - I_z}{I_x}$, $a_2 = \frac{-J_r}{I_x}$,$a_3 = \frac{I_z - I_x}{I_y}$,$a_4 = \frac{J_r}{I_y}$,$a_5 = \frac{I_x - I_y}{I_z}$,$b_1 = \frac{d}{I_x}$,$b_2 = \frac{d}{I_y}$, $b_3 = \frac{d}{I_z}$.

Then, the criterion is formulate as follows:

$$J = \rho \int_0^{t_f} (u_1^2 + u_2^2 + u_3^2 + u_4^2) dt \tag{42}$$

where $t_f(second)$: free final time.

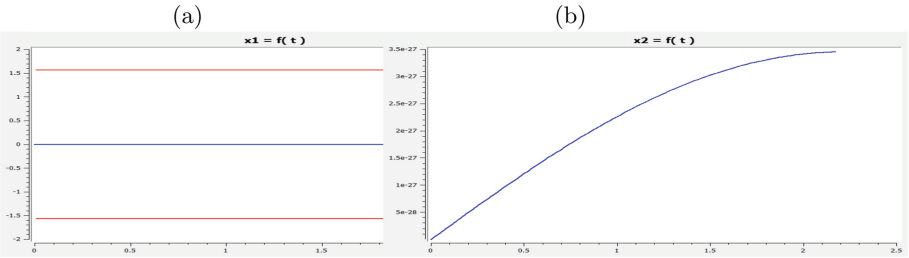

Fig. 2. Trajectory state of x_1 and x_2 respectively. (Color figure online)

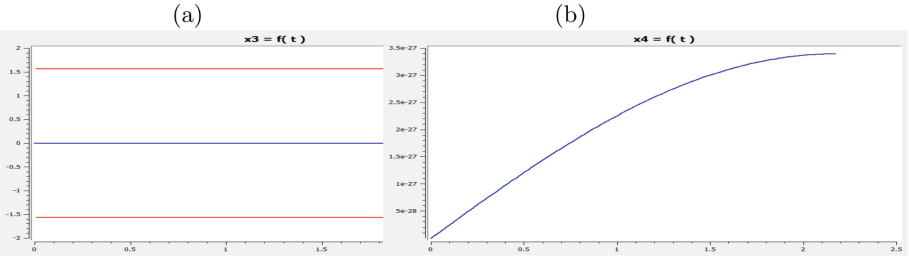

Fig. 3. Trajectory state of x_3 and x_4 respectively. (Color figure online)

4 Simulation and Discussion

$\omega = 340; d = 0.23;\ m = 0.6;\ I_x = I_y = 7.5e-3,\ I_z = 1.3e-2;\ J_r = 6e-5;\ g = 9.8.$

The red line is the delimiter of $x_1,\ x_3,\ x_5$ respectively. And the blue line is the trajectories.

The results given by Bocop software are presented in Figs. 1, 2, 3, 4, 5, 6, 7 and 8. And Figs. 9 and 10, is the criterion for different values of ρ. When $10^{-5} < \rho < 1$ and,$10^{-2} < \rho < 1$, the minimum is not reached. Else, the optimal solution is ensured in 7 iterations with 1.41 s, and final time $t_f = 2.17214$ s (Fig. 11).

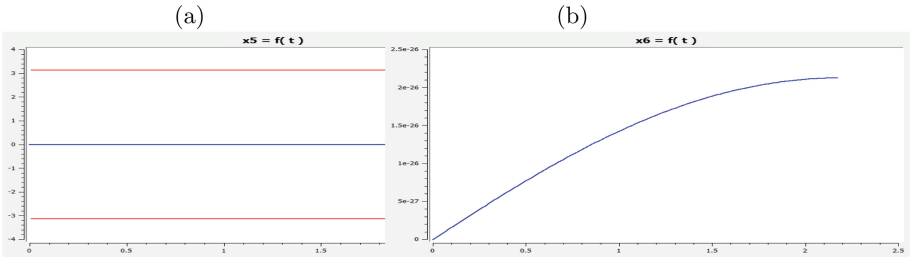

Fig. 4. Trajectory state of x_5 and x_6 respectively. (Color figure online)

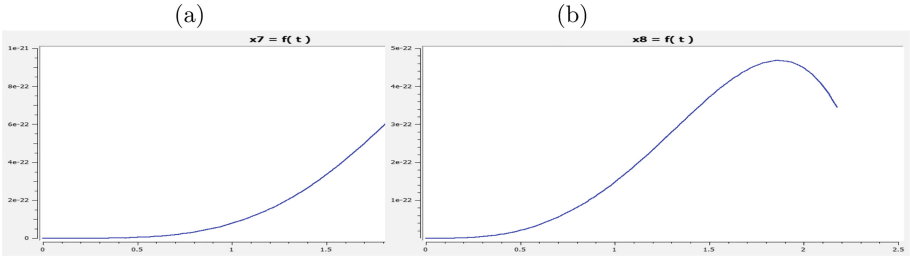

Fig. 5. Trajectory state of x_7 and x_8 respectively.

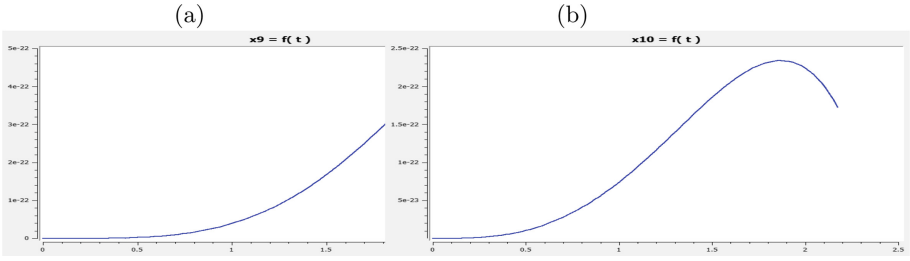

Fig. 6. Trajectory state of x_9 and x_{10} respectively.

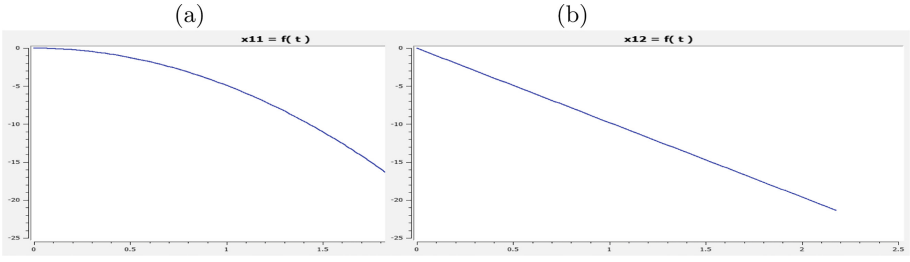

Fig. 7. Trajectory state of x_{11} and x_{12} respectively.

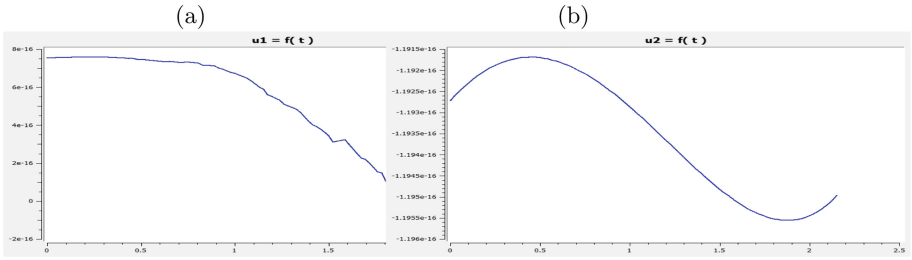

Fig. 8. Control u_1 and u_2 respectively.

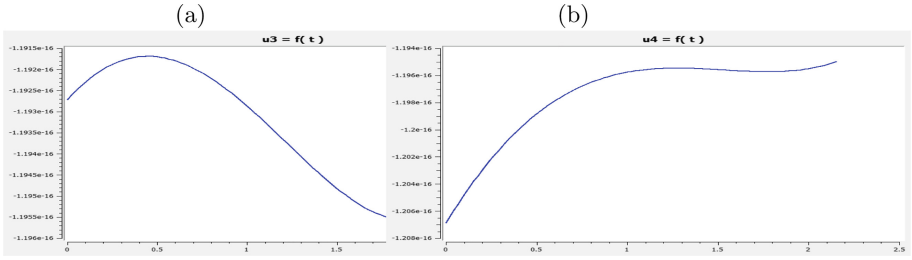

Fig. 9. Control of u_3 and u_4 respectively.

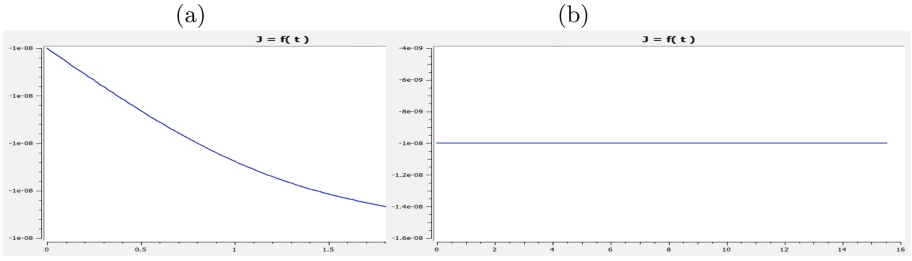

Fig. 10. The criterion J of $\rho = 0.0002$ and $\rho = 0.002$ respectively.

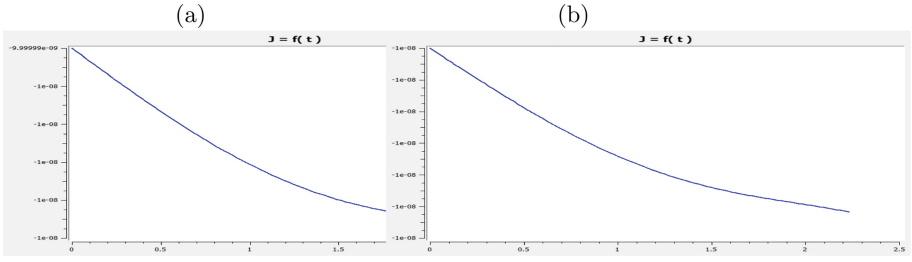

Fig. 11. The criterion J of $\rho = 0.0004$ and $\rho = 0.0005$ respectively.

5 Conclusion

In this work, we have solved an optimal control problem of unmanned aerial vehicle to minimize the energy of quadrotor in free final time.

The results are adequate for our purpose in the computational time is 1.41 s in 7 iterations with Bocop software. The convergence is fast and the computational time is small.

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